A GRAPH THEORETIC APPROACH TO CAD FOR THE ANALYSIS OF PLANAR MECHANISMS

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ABSTRACT

One of the main goals of multibody dynamics research is to develop formulations that automatically derive the equations of motions for complex mechanical systems. Equations of motion are formulated systematically and then solved numerically. This paper studies analytic methods for formulating the equation of motions, with emphasis on the graph theoretic modelling approach. The graph theoretic approach is proposed for modelling and analysis of planar mechanical systems. The development of the CAD system incorporating graph theoretic models would be used for mechanical analysis. A graphical user interface will be used to capture the design and to display the solutions of the dynamic and kinematic computations. Throughout the analysis, the CAD tool (PMech) would be useful for relieving the engineer's task of doing the calculations by hand, the result of kinematic and dynamic analysis of the different prototypes constitutes the graphical display of the mechanism, enhancing the engineer's understanding of the mechanical behaviour under consideration.

Keywords : CAD, computer graphics, constraint motion, graph theory, simulation.

1. INTRODUCTION

Planar mechanical systems are systems in which translation occurs only in two dimensions. They typically include linkages, rigid bodies, joints, springs, dampers and external influences. An graph theoretic approach can be used for modeling and analysis of planar mechanical systems. The graph theoretic approach brings generality and allows a systematic technique for analysis and design to be developed. Planar systems as a class of mechanism are of interest for a number of reasons:

- *Pedagogical*. Planar systems are easier to visualize and hence to understand.
- *Simpler representation of bodies*. In motion restricted to planar translation and rotation about the perpendicular axis, the position of a body can be represented by three variables (x,y,q).
- *Treatment of angular orientation*. For planar systems, rotation is a vector, while in spatial systems, rotation is not a vector quantity. This means that spatial systems must use angular velocity or acceleration which are vector quantities or use a different representation of angular orientation, such as Euler angles.

The treatment of linear systems using graph theoretic modelling is well developed and textbooks on the subject are available [Chen96, Koeni67].

Planar systems or subsystems are encountered often enough in practical applications to warrant special treatment as a separate class, just as one dimensional motion can be studied separately as they are usually linear systems, and can be studied in the Laplace domain. Planar systems are given separate treatment by several authors [Nikra88, Roger75] in textbooks and research papers

2. GRAPH THEORETIC MODELS

In order to apply formal graph theoretic methods to a multibody system [Koeni67], the concept is to form a linear graph that describes the system, or encapsulates the system topology. The formation of the system graph is accomplished by identifying the significant points in the physical system, and

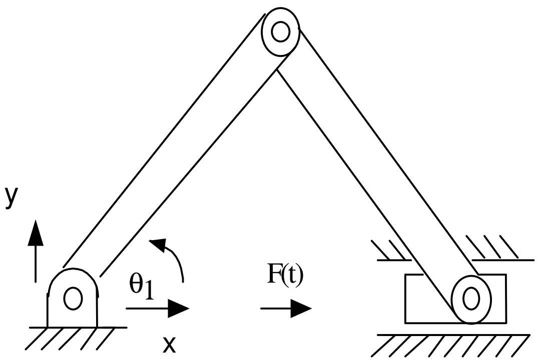

Figure 1: The slider crank mechanism

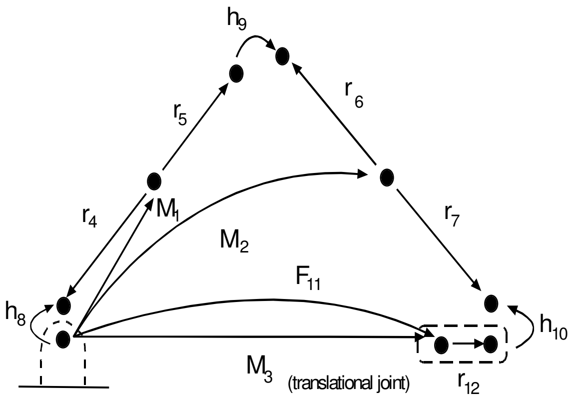

Figure 2: System graph of the slider crank mechanism

representing them by the vertices (or nodes) of a graph. For mechanical system, these vertices would correspond to centres of mass for rigid bodies, points on a body to which forces are applied or joints are connected, and a ground that represents the origin of an inertial reference frame.

The graph theoretic model of a component is the terminal representation that consists of a terminal graph (or edge) and a set of terminal equations that relate the complementary variables of the component measured at the component terminals. There are two generic types of variables: across: variables and through: variables. In the translational subspace the across variables are displacements, velocities and accelerations, while force is the through variable. In rotation, angular displacements, velocities and accelerations are the across variables, while torque is the through variable.

The graph theoretic approach implies that the formulation process involves the formation of a system graph that represents the actual mechanisms the engineer is trying to analyse. A system graph generally consists of a translational graph and a rotational graph. For example, a slider-crank mechanism of Figure 1 is represented by a translational graph of Figure 2. Thereafter, all the formulations techniques developed are based on the derived system graph accordingly. While formulation of the mechanism using the direct description of the structure will continue to be very important, the research effort described in this paper discusses the use of the graph theoretic method for formulating the equations of motions of the planar physical system. Formulation methods can be applied to reduce the number of equations that need to be solved simultaneously.

For complex mechanical systems, this generally involves the numerical integration of a set of second order differential-algebraic equations. While explicit forms may be possible in some instances, it may not be practical to obtain them. The equations to be solved have additional characteristics of being stiff, nonlinear, and implicit.

The topology of a graph is described by the various matrices associated with that graph, and the topology of a graph has two postulates associated, they are known as the Vertex and Circuit Postulates. The Vertex Postulate means that the sum of the through variables associated with the edges incident away from any vertex in a system graph is equal to the sum of the through variables associated with the edges incident towards that vertex. The vertex postulate is analogous to the Kirchoff Current Law in the Electrical Engineering domain.

The Circuit Postulate means that the sum of across variables associated with the edges that are in any circuit and that have the assigned circuit orientation is equal to the sum of the across variables associated with the edges that are in the circuit and have the opposite orientation. The Circuit Postulate is analogous to Kirchoff Voltages Law. The Vertex Postulate implies a set of force balanced equations:

$$\mathbf{A}\vec{\mathbf{F}} = 0 \text{, the vertex equation} \tag{1}$$

where $\mathbf{A}$ is the incidence matrix of the system graph, and $\vec{\mathbf{F}}$ is a vector of through variables. Since the fundamental cutset matrix of the system graph carries the vertex postulate information, the following equation is valid and is known as the fundamental cutset equation:

$$\mathbf{Q}_\text{f}\vec{\mathbf{F}} = 0 \tag{2}$$

The f-cutset matrix is denoted by $\mathbf{Q}_\text{f}$. Furthermore, by partitioning the set of through variables into tree $\vec{\mathbf{F}}_t$ and cotree $\vec{\mathbf{F}}_c$ forces, one can use the cutset equations to relate the two:

$$\left[\mathbf{U} \middle| \mathbf{Q}\right]\begin{bmatrix}\vec{\mathbf{F}}_\text{t}\\ \vec{\mathbf{F}}_\text{c}\end{bmatrix} = 0 \tag{3}$$

$$\vec{\mathbf{F}}_t = -\mathbf{Q}\vec{\mathbf{F}}_c \tag{4}$$

also

$$\vec{\mathbf{F}}_c = \mathbf{U}\vec{\mathbf{F}}_c \tag{5}$$

Writing these two equations together:

$$\begin{bmatrix} \vec{\mathbf{F}}_t \\ \vec{\mathbf{F}}_c \end{bmatrix} = \begin{bmatrix} -\mathbf{Q} \\ \mathbf{U} \end{bmatrix} \vec{\mathbf{F}}_c \tag{6}$$

or

$$\begin{bmatrix} \vec{\mathbf{F}}_t \\ \vec{\mathbf{F}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{B}^T \\ \mathbf{U} \end{bmatrix} \vec{\mathbf{F}}_c \tag{7}$$

$$\vec{\mathbf{F}} = \mathbf{B}_f^T \vec{\mathbf{F}}_c \tag{8}$$

The chord transformation matrix is the transpose of the circuit matrix. Equation (8) is known as the "chord transformations". As the terminal equations alone are insufficient in number to determine the time response of a given mechanical system, the topological equations representing the connectivity of the system components are required. The cutset equation (4) represents a set of linearly independent equations, one for each of the branches. Hence the derivation of the chord transformation begins by the use of the cutset equation.

Similarly, the circuit postulate implies a set of equations relating the across variables. From the fact that the fundamental circuit matrix carries exact information about the circuit postulate and equation (5) is known as the fundamental circuit equation:

$$\mathbf{B}_f \vec{\mathbf{R}} = 0 \tag{9}$$

where $\mathbf{B}_f$ is the circuit matrix, and $\vec{\mathbf{R}}$ is a vector of across variables. The circuit matrix carries information about a vector loop equation during the existence of a mechanical joint (a revolute joint), hence the equation forms the relevant constrained equations for the case of constrained mechanical system typically encountered in the problem domain. The circuit equations represent a set of linearly independent equations, one for each chord in the cotree. Hence similar to the chord transformations, a set of "branch transformation" equations can be obtained by partitioning the circuit equations. From which we can write:

$$\begin{bmatrix} \mathbf{B} \mid \mathbf{U} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{R}}_t \\ \vec{\mathbf{R}}_c \end{bmatrix} = 0 \tag{10}$$

$$\vec{\mathbf{R}}_c = -\mathbf{B}\vec{\mathbf{R}}_t \tag{11}$$

Thus, one only needs to determine a set of across variables in a tree in (11), a set of through variables in the cotree in (4), and the rest can be computed from the derived transformation equations. The method considered has a convention that force drivers will be placed in the cotrees. By further appending the circuit equation (11), we can summarize the branch transformation as :

$$\begin{bmatrix} \vec{\mathbf{R}}_t \\ \vec{\mathbf{R}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ -\mathbf{B} \end{bmatrix} \vec{\mathbf{R}}_t \tag{12}$$

or

$$\begin{bmatrix} \vec{\mathbf{R}}_t \\ \vec{\mathbf{R}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ \mathbf{Q}^T \end{bmatrix} \vec{\mathbf{R}}_t \tag{13}$$

$$\vec{\mathbf{R}} = \mathbf{Q}_f^T \vec{\mathbf{R}}_t \tag{14}$$

3. SYSTEMATIC FORMULATIONS OF MOTION EQUATIONS

To formulate dynamic equations for the mechanical system, the graph theoretic method requires several steps to be completed. The first step is the generation of the cutset equations from the graphs [McPhe97]. For a mechanical system containing closed kinematic chains, the circuit equations are used. The branch and chord transformations is the next step of the procedure to eliminate any variables. Assuming that all the constraints are holonomic, the second and the third step results in a set of nonlinear algebraic equations, representing the kinematic structure of a constrained mechanical system:

$$\mathbf{\Phi}(q,t) = 0 \tag{15}$$

The generalized coordinates q can be obtained by solving the equation using an appropriate numerical method, such as the Newton-Raphson procedure.

Finally, if a dynamic analysis is required, the constraint equations are insufficient in number to solve for $q(t)$. In this case, the constraint equations are assembled into the differential equations produced after steps 1 to 3:

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{g} \tag{16}$$

where $\mathbf{M}$ is a symmetric positive-definite mass matrix and $\mathbf{g}$ contains external loads and the weight of the rigid bodies, should the gravity factor is considered together, $\boldsymbol{\lambda}\,(\mathbf{t})$ is the Lagrange multiplier representing the joint reaction force for a constrained planar mechanical system.

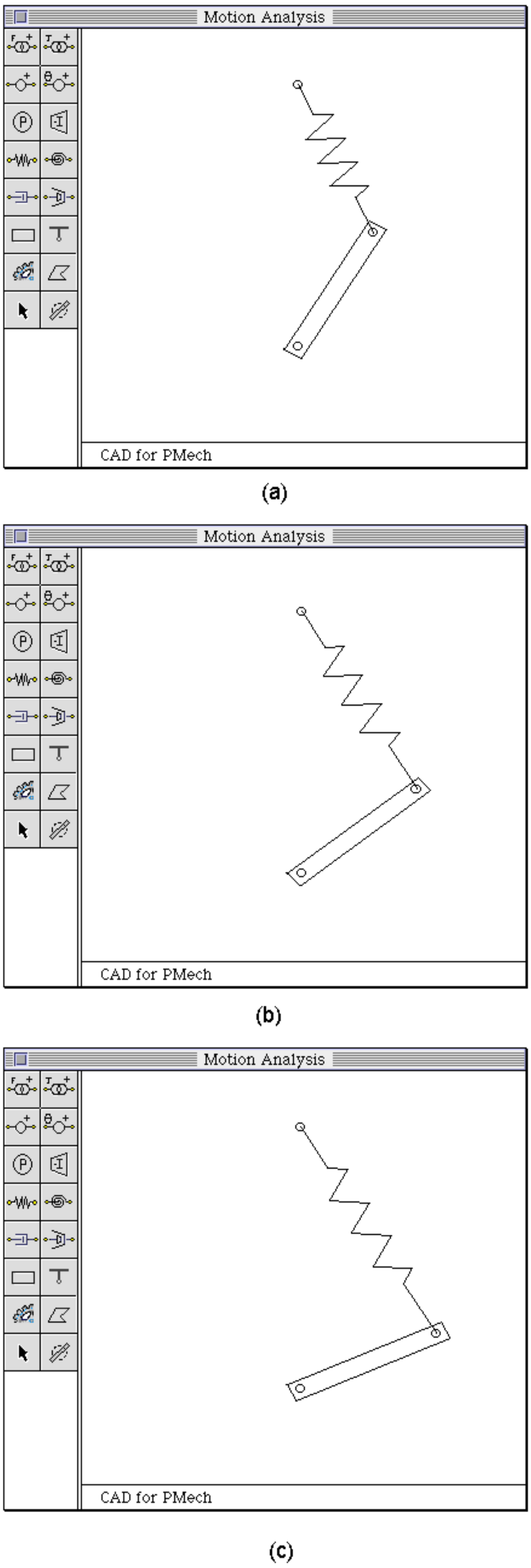

Figure 3: The CAD interface for visualization

4. THE CAD SYSTEM FOR ANALYSIS

A component palette of Figure 3 is provided for the selection of components and ease of description of the mechanisms that the designer may have in his mind. When the system solutions are evaluated successfully, the solutions would be used by the CAD system to visualize the system dynamics. An example of a spring-suspended pendulum platform is shown in Figure 3. The user interface is modelled after 2D CAD systems commonly used in mechanical design. A palette predefined types of components is provided to allow the user to create individual instances. The user also specifies the interconnection between components. The component data and the topology is captured by the interface for use by subsequent analysis stages. A CAD like interface is used to facilitate description of the system being designed by the user to the analysis program. The usr of a CAD like interface allows a user to specify geometry in conjunction with topology. This is in contrast to design of electrical systems where design can be done at the functional level largely independent of the geometry of the components and system. A CAD like interface also provides a more intuitive design interface for mechanical design due to similarity to other mechanical CAD systems.

In the display of simulation results, the users should have control of how the results are displayed (e.g. the frame rate, integration time step), and what components or nodes are selected for inclusion. A graphical user interface is used to capture the design and to display results of simulation. Device may be attached to particular components to display plots of the solutions or tables of results.

5. ANALYSIS OF SYSTEMS DYNAMICS

The cutset, circuit, and terminal equations together constitute a necessary and sufficient set of equations for representing the motion of a mechanical system. Due to its systematic nature, the formulation was encoded into a sequence of Maple procedures. One advantage of using a commercial symbolic package is that the velocity and acceleration equations can be directly obtained by symbolically differentiating the constraint equations with respect to time; this greatly reduces the amount of programming effort required [McPhe97].

For a constrained mechanical system with m independent constraints. Equation (17) is used for position analysis of any planar mechanical system.

$$\mathbf{\Phi} = 0 \tag{17}$$

where $\mathbf{\Phi}$ is a set of nonlinear algebraic equations for the constrained system [Nikra88]. After performing the first and second time derivative of $\mathbf{\Phi}$, we obtain the velocity and acceleration constraint of the system:

$$\mathbf{\Phi}_q \dot{\mathbf{q}} = 0 \quad (18)$$

$$\mathbf{\Phi}_q \ddot{\mathbf{q}} - \boldsymbol{\gamma} = 0 \quad (19)$$

where $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ is the generalized velocities and accelerations of the system respectively. The Lagrange's equations of motion for a constrained system should be assembled into this form:

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{g} \quad (20)$$

The initial conditions for the equations of motion at time $t = t_0$ are specified as in equation (21). The initial position $\mathbf{q_0}$ and velocity $\dot{\mathbf{q}}_\mathbf{0}$ must be consistent with system constraints. The initial positions and velocities can be selected as functions of design variables to generate the desired dynamic response for an optimal design problem. For constrained mechanical systems, equations (15) - (17) must be considered together as the system equations of motion. Following the graph theoretic approach, the projected cutset and circuit equations should be assembled into the standard matrix form of equations of motions of (22).

$$\left.\begin{aligned} q(t_0) &= q_0 \\ \dot{q}(t_0) &= \dot{q}_0 \end{aligned}\right\} \quad (21)$$

$$\begin{bmatrix} \mathbf{M} & -\mathbf{\Phi}_q^T \\ \mathbf{\Phi}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{bmatrix} \quad (22)$$

The Jacobian matrix $\mathbf{\Phi}_q$ is a function of $\mathbf{q}$, and vectors $\mathbf{g}$ and $\boldsymbol{\gamma}$ are functions of $\mathbf{q}$, $\dot{\mathbf{q}}$, and t. Therefore, at any given instant, if $\mathbf{q}$ and $\dot{\mathbf{q}}$ are known, the equation provides the set of DAE for the unknowns $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$ that can be solved for. A number of methods have been developed in the literature to solve a set of mixed Differential-Algebraic Equations (DAE). The performance of various other approaches such as the backward difference formulas, generalized coordinate partitioning, constraint stabilization method would be further investigated.

A simple direct integration scheme has been developed in this project, where the position and velocity variables are appended together as a vector in a state space form:

$$\mathbf{y} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \quad (23)$$

The velocity and acceleration vectors are represented in the vector:

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} \quad (24)$$

At time $t = t_n$, the vector $\dot{\mathbf{y}}_i$ can be integrated numerically to obtain $\mathbf{y}_{i+1}$, where $t_{i+1} = t_i + \Delta t$.

An example system has been analysed. A planar view of the system in Figure 4 is connected to ground by its bodies 2 and 5 at joint A and E respectively. The mass and moment of inertia of each body is 0.6 and 0.1 respectively. There is a spring element with parameters $k = 60$, $l_0 = 7$. When the angle of the leg of body 2 and 5 with the horizontal is $110°$, the system is released from rest. Kinematic analysis is performed for the initial conditions for $\mathbf{q}$ and $\dot{\mathbf{q}}$. The displacement solution of the x coordinate of the body 3 of the system in Figure 5 provides the information concerning the motion of the planar system over time. The spring is arranged in a way that it lightly pushes the system to the left initially, as it is slightly compressed when the system is released. The displacement solution in Figure 5 shows that it conforms to the expected behaviour in way that the x value is decreasing, indicating the structure is rotating to the left.

6. CONCLUSION

The computer implementation of the project contains two aspects. The formulations were successfully encoded into a sequence of the Waterloo Maple V procedure, and the numerical results have been carefully evaluated so as to verify its robust operation and the validation of the formulation methods. Algorithms have been devised to obtain the constrained algebraic equations directly from the system graph and they are subsequently adopted for the construction of the DAE for further integration process.

A second computer implementation was developed, using an object-oriented programming language (C++), and the resulting software (PMech, for planar mechanical systems analysis) is designed to run on an Apple Macintosh computer with release 8.0 of the Macintosh Operating System. One interesting feature of this software is the development of a computer interface for direct manipulation of selectable mechanical components. The computer interface would be suitable for multibody system design, and the visualization of motion behaviour of the system under analysis. The development of the CAD interface makes use of the several thousands APIs, which are defined by the original computer manufacturer and offered by other commercial high-level language compilation system (CodeWarrior Pro 2.1) used in the project.

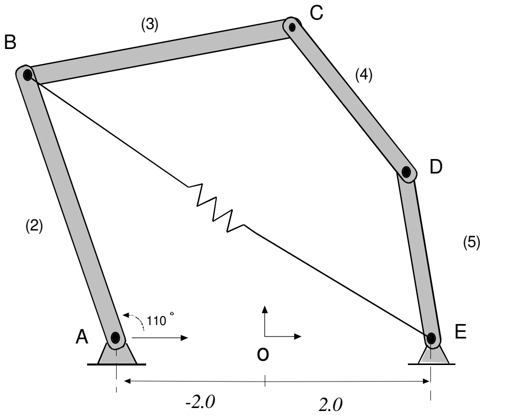

Figure 4: A planar mechanism

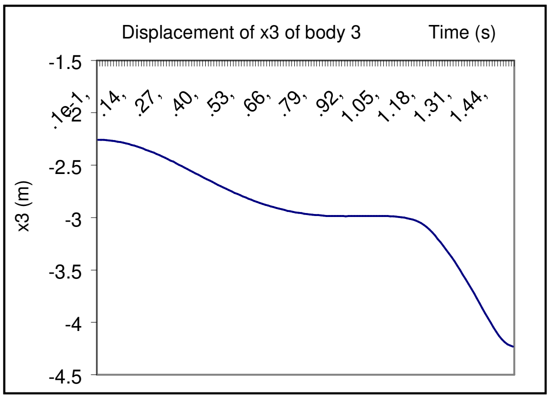

Figure 5: The displacement *x*3 of planar system

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