MATHEMATICAL FREE-FORM SOLID MODELLING BASED ON EXTENDED SIMPLICIAL CHAINS

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ABSTRACT

A mathematical model for geometric modelling is presented in this work. This model is based on the concept of *extended simplicial chain*. Solids are defined by means of simple algebraic operations (additions and subtractions) performed on what we call *extended cells*. In like manner, by using the operations defined for extended simplicial chains, we obtain the traditional Boolean operations in geometric modelling. Thus, we define a new model which allows us to represent and operate with free-form solids, particularised for the case of solids whose boundaries are free-form surfaces expressed as a set of low degree algebraic patches. However, the concept of extended simplicial chain is more general and wide both in the dimension in which the solid is defined and in the type of surface used for its representation.

Keywords: formal methods in computer graphics, mathematical models, free-form solid modelling, Boolean operations.

INTRODUCTION

Counting on mathematical models as formal substratum of a particular geometric modelling method provides unquestionable advantages [Duce91]: homogeneous study of the representation systems properties and easy implementation. In this work, we present a mathematical model that allows us to represent and operate with *d*-dimensional free-form solids, developed in 3D for solids whose boundaries are expressed as a set of algebraic patches. There are two fundamental elements needed to make up our formalism; firstly the previous *simplicial chains* [Feito98] and secondly the called *free-form cells*, noted *ffc*, derived from the use of curved surfaces in solid definition (in this work the surfaces are *algebraic patches* [Seder85, Guo91, Menon94, Bajaj95]). By joining these elements, we define the concept of *extended simplicial chain*, this is the basis for representing both free-form solids and the operations with them. In this way, we obtain a new mathematical model to deal with free-form solids. Based on this scheme it is very easy to developed fundamental algorithms in solid modelling such as simple and robust point-in-solid tests in both two and three dimensions [Ruiz97, Ruiz99].

The remainder of the paper is organised as follows: first we review the concepts and problems in similar previous works. Afterwards we introduce the basic concepts of the base elements of our formalism, *free-form cells* (*ffc*), studying the particular case of the *ffc* defined by an algebraic patch; later on we introduce the concept of *extended simplicial chain*, and finally we show how free-form solids and Boolean operations can be represented by means of extended simplicial chains.

PREVIOUS WORK

There are many works that formalise the representation and manage of free-form solids [Shapi91, Menon94, Menon96]. The most interesting that works with solids whose boundaries are defined by algebraic patches is the presented by Dr. Menon [Menon94]. His method, from a mesh of algebraic patches, which describe the solid surface, makes a CSR (*Constructive Shell Representation*) of the solid as the union of the *trunctets* generated by each algebraic patch. From this *CSR* and the *input polyhedron* of the solid an exact CSG representation of the solid is established [Menon96]. The method developed to obtain this exact CSG representation is more complicated when appear the called *abnormal constructions*, basically when the intersection of the *trunctets* is not empty (see figure 1). These situations are solved by using techniques such as *separation half-spaces* [Shapi93], whose calculation in many cases is difficult and computational expensive, and most algorithms need this CSG representation to work. An example of this situation is shown in figure 1, furthermore, this example presents the additional problem of an *input polyhedron* with self-intersection.

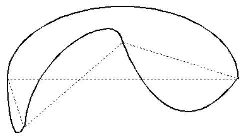


Figure 1: *Abnormal construction that needs a special deal.*

In the next section, we introduce the base element of the proposed formalism, that is, the concept of free-form cell, specified for the case of algebraic patches. From this new element, we generate the concept of extended simplicial chain, an alternative way to represent free-form solids avoiding to deal with particular cases and including models as the shown in figure 1.

FREE-FORM CELL

The main goal in this work is to model formally solids with free-form boundary, that is, the boundaries are defined by elements of free nature called *free-form elements* (curves in two dimensions, surfaces in three dimensions,...). This property requires introducing concepts into the formalism in order to adequate, simple and homogeneous deal with the non-planar zones of solids. In this sense, this section presents the free-form equivalent to planar simplex (triangle, tetrahedron,...) used in [Feito98]. This new element is called *free-form cell*. Firstly we see the general concept, dimension and surface type independent, and afterwards, we focus in free-form cells defined by an algebraic patch, case of study of this work.

General free-form cell

Before a formal definition, we can see a *free-form cell* as the closed set of points (in the dimension we are working) limited by one free-form element and several *planar elements* (line segments in 2D, polygons in 3D,...). The planar elements that bound a free-form cell depend on the particular free-form element used to define it, and they have the property of be common to other simplices or free-form cells used to define the solid. In the remainder of the paper, when we want to refer to any of the elements which bound a free-form cell without taking into account if it is planar or free-form, we use the term *bound element*. Next definition formalises the intuitive concepts saw.

Definition 1. *Free-form cell* (*ffc*) is the set of points in $\mathbb{R}^d$ obtained as the intersection of the half-spaces defined by a free-form element of dimension d and by one or several planar elements of dimension d-1 that holds:

a) the set of points defines a closed and connected set in $\mathbb{R}^d$,

b) all points in the set, less the boundary, lie on the same connected component (having equal signs) with regard to the implicit equation of the free-form element, and

c) the points common to a number of bound elements greater than or equal to d belong to the free-form element too. ◆

Condition a) eliminates configurations as figure 2.a, which can be split in simpler *ffc*'s as figure 2.b. Condition b) is needed because the set of points can define a closed and connected set but where not all the points have the same sign with regard to the implicit equation of the free-form element (see figure 3.a). However, this situation supposes no limitation because we can split this set of points (more or less easily) in order to hold condition b) (see figure 3.b).

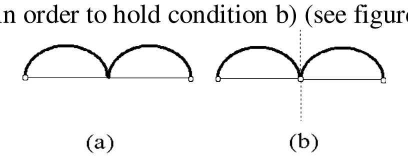


Figure 2: *Condition a) of a ffc.*

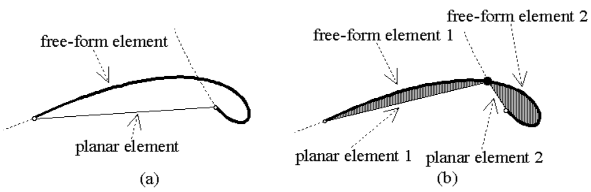


Figure 3: *Condition b) of a ffc.*

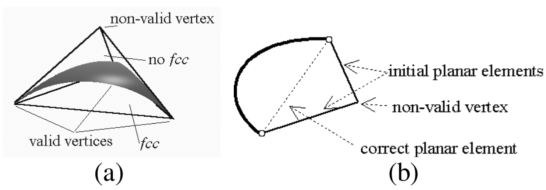


Figure 4: *Condition c) of a ffc.*

Condition c) (trivially verified when any of the bound elements is the free-form one of the *ffc*) gives a method to identify the vertices of the *ffc* (see figure 4.a), discriminates between different candidates to *ffc* which verify conditions a) and b) (see figure 4.a) and eliminates complex configurations easy to simplify (see figure 4.b).

Each *ffc* has an *associated sign* of value 1 or -1. This sign gives an information about the orientation of the *ffc* with regard to the solid, and it has a fundamental roll in definition of both solids and operations with them. When free-form cells defined by algebraic patches be studied, then we will give accurate methods to calculate the signs of this particular *ffc*. There exist several methods to calculate the sign of bidimensional *ffc*'s [Ruiz97].

Free-form cell defined by an algebraic patch

In three dimensions, the free-form element used to define a *ffc* is a surface. The concept of *ffc* has no limitation with regard to the type of surface used in its definition, but the use of a particular surface supposes both the study of conditions to hold definition 1 and to carry out methods in order to determine the relative position of a point with regard to the *ffc* defined by this surface. In this work, we use algebraic patches to construct the boundary of the solid.

Definition 2. The *algebraic patch* $p(f_T)$ is defined over the tetrahedron T as the zero contour inside T of function $f_T = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha_T)$, where n is the degree of the algebraic patch, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a nonnegative integer tuple, B are Bernstein polynomials, and b_λ is the weight of the control point λ (see figure 5). That is, if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the barycentric coordinates of $\mathbf{Q}$ with regard to T:

$$p(f_T) = \{\mathbf{Q} \,/\, f_T(\mathbf{Q}) = 0 \wedge \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0\}. \quad ◆$$

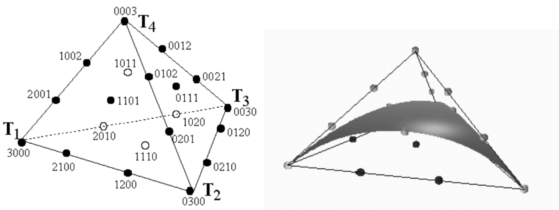
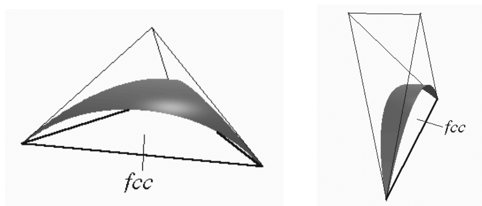


Figure 5: *A cubic algebraic patch and its mesh of control points.*

The method to construct the surface that defines the free-form solid boundary is the following: we start from an *initial triangulation* of the surface of the solid whose vertices and normals at vertices must be interpolated by the final surface. For each triangle, a new vertex is associated, generating both a tetrahedron and the appropriate algebraic patch inside it, called 3-*patch*. This is carry out by using methods [Guo91, Bajaj95] that obtain unique zero contour without singularities inside the tetrahedron. The two-sided holes [Guo91] generated between each pair of patches are filled with appropriate blend-patches [Guo91, Bajaj95], called 1-*patch* (see figure 6.b). There are detailed methods [Dahme89, Guo91, Bajaj95] to generate these algebraic patch meshes. In this way, a final surface that smoothly interpolates the initial triangulation is obtained. Therefore, we can introduce the concept of *ffc* defined by an algebraic patch with the next definition.

Definition 3. Let $p(f_T)$ be an algebraic patch which defines a single sheeted without singularities inside the tetrahedron T. The *free-form cell defined by* $p(f_T)$ is that whose free-form element is $p(f_T)$ and whose planar elements are the four faces of T. ◆

Figure 6 shows the *ffc*'s defined by the two types of algebraic patches in the boundary of a free-from solid (3-patches and 1-patches). Conditions a) and b) in definition 1 are automatically held because we are using the particular algebraic patches [Guo91, Bajaj95] generated in the initial triangulation interpolating process. Condition c) in definition 1 selects the only *ffc* between the two half-spaces defined by an algebraic patch.



(a) (b)

Figure 6: *ffc's defined by algebraic patches.*

Therefore, two are the particular types of *ffc* defined by algebraic patches to take into account; that defined by a 3-patch, called *free-form cell 3ffc* (fig. 6.a), and that defined by a 1-patch, called *free-form cell 1ffc* (fig. 6.b). All *ffc* have an associated sign of value 1 or -1. The next definitions establish how to calculate this sign for free-form cells 3*ffc* and 1*ffc*.

Definition 4. Let $p(f_T)$ be a 3-patch, generated in the initial triangulation interpolating process, and let $3ffc_p$ be the *ffc* that it defines. The *sign of* $3ffc_p$ is calculated as the sign (+1 if positive, -1 if negative) of the weight b_{000n} of $p(f_T)$, that is, the sign of the weight of the control point $\mathbf{T}_4$. ◆

Definition 5. Let $ps(f_{T1})$ be a 1-patch, generated in the initial triangulation interpolating process that is associated with the 3-patch $pp(f_{T2})$ to fill a two-sided hole; let $1ffc_{ps}$ be the *ffc* defined by $ps(f_{T1})$ and let $3ffc_{pp}$ be the *ffc* defined by $pp(f_{T2})$. The *sign of* $1ffc_{ps}$ is equal to the *sign of* $3ffc_{pp}$. ◆

A 1-patch is generated in the process of filling two-sided holes between two 3-patches, and therefore, the sign of the *ffc* that it defines is equal to the sign of the *ffc* of the 3-patch with which it is associated to fill a particular two-sided hole. Intuitively, a 3-patch (1-patch) has a sign of value 1 if the surface patch face looking to the base triangle (base edge) is touching the interior of the solid; and the sign is -1 if this surface patch face is touching the exterior of the solid. The methods [Guo91, Bajaj95] that construct free-form surfaces assign the weight b_{000n} in this way. Finally, we have all ingredients to study the concept of extended simplicial chain.

EXTENDED SIMPLICIAL CHAINS

Using *ffc*'s, we can extend the concept of *simplicial chain* introduced in [Feito98] as follows:

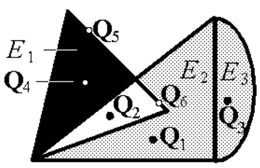
Definition 6. The *d-dimensional extended simplicial chain* δ is defined as the expression:

$$\delta = \sum_i \alpha_i E_i$$

where E_i is an *extended cell* (simplex or *ffc*) of dimension d and α_i is its coefficient, an integer of value -1, 0 or 1. ◆

When a simplex has any of its vertices on the origin, then we call it an *original simplex*. Similarly to the case of *ffc*, the simplices have an associated sign of value 1, -1 or 0. In two dimensions, the sign of a triangle (2-simplex) is calculated as the sign of the signed area of the triangle. In three dimensions, the sign of a tetrahedron (3-simplex) is calculated as the sign of its signed volume [Feito98].

Over these chains the sum operation (adding the elements of each chain) and the product of a chain by a scalar (by multiplying each of the chain coefficients by the scalar) are defined. An extended simplicial chain has an associated function that is the basis to identify the free-form solid represented by the chain.



$\delta = -E_1 + E_2 + E_3$

$f_\delta(\mathbf{Q}_1) = 0 + 1 + 0 = 1; f_\delta(\mathbf{Q}_2) = -1 + 1 + 0 = 0$

$f_\delta(\mathbf{Q}_3) = 0 + 0 + 1 = 1; f_\delta(\mathbf{Q}_4) = -1 + 0 + 0 = -1$

$f_\delta(\mathbf{Q}_5) = -1 + 0 + 0 = -1; f_\delta(\mathbf{Q}_6) = -1 + 1 + 0 = 0$

Figure 7: *Extended simplicial chain in 2D.*

Definition 7. Let δ be an extended simplicial chain. We define the *associated function of the chain* δ, noted f_δ, as:

$$f_\delta: \mathbb{R}^d \rightarrow \mathbb{Z}$$

$$f_\delta(\mathbf{Q}) = \sum_{\mathbf{Q} \in E_i} \alpha_i$$ ◆

Taking into account that $\mathbf{Q}$ is a point (in the plane if d=2, in the space if d=3...), then f_δ is equal to the sum of the coefficients of the extended cells that contain $\mathbf{Q}$. Figure 7 shows an example of extended simplicial chain in 2D and some values of its associated function.

We immediately notice that we can associate a free-form solid to each extended simplicial chain, the one defined by the points for which the function associated with the chain returns a value different from zero (shaded zones in figure 7).

Definition 8. We define the *free-form solid associated with the d-dimensional extended simplicial chain* δ, noted FF_δ, as:

$$FF_\delta = \{\mathbf{Q} \in \mathbb{R}^d \,/\, f_\delta(\mathbf{Q}) \neq 0\}.$$ ◆

A fundamental question to consider is the regularisation of operations. If we do not consider this question, then problems can arise in zones common to extended cells with coefficients of opposite signs (see point $\mathbf{Q}_6$ in figure 7). For this reason, we re-define the concept of associated function of a chain in the following manner.

Definition 9. (Associated regular function of an extended simplicial chain). Let δ be an extended simplicial chain, let $f_\delta(\mathbf{Q})$ be its associated function and FF_δ the associated free-form solid of the chain δ. We define the associated regular function of the chain δ, noted $f_\delta^*(\mathbf{Q})$, as:

$f_\delta^*(\mathbf{Q}) = 0$, if $Q \notin cl(int(FF_\delta))$
$f_\delta^*(\mathbf{Q}) = p$, if $p > 0$
$f_\delta^*(\mathbf{Q}) = n$, otherwise

where $p = \max\{f_\delta(\mathbf{Q}') \,/\, \mathbf{Q}' \in \varepsilon(\mathbf{Q})\}$, $n = \min\{f_\delta(\mathbf{Q}') \,/\, \mathbf{Q}' \in \varepsilon(\mathbf{Q})\}$, *cl* is the closure operation, *int* represents the interior points and ε is a neighbourhood small enough. ◆

In this way, now, in figure 7, $f_\delta^*(\mathbf{Q}_6)$=1 and $f_\delta^*(\mathbf{Q}_5)$ = -1. We can now define regular operations between extended simplicial chains.

Definition 10. Let δ and δ' be two extended simplicial chains and λ a scalar value. The associated regular functions for the sum of the extended simplicial chains δ and δ', and for the product of λ by δ are respectively expressed as:

$f_\delta +^* f_{\delta'} = f^*_{\delta+\delta'}$
$\lambda \cdot^* f_\delta = f^*_{\lambda \cdot \delta}$ ◆

Thus, the regular operations have, for extended simplicial chains, the same properties as the non-regular operations; that is, they have a vectorial space structure [Feito98]. From now on, we assume associated functions and operations to be regular. In next section, we show how to calculate an extended simplicial chain that represents a particular free-form solid.

FREE-FORM SOLID REPRESENTATION

From definition 8, we deduce that different chains can have the same associated free-form solid. Therefore, we must establish some uniform criterion that, given a free-form solid, selects one representative chain among all the possible chains. These chains are called *normal* extended simplicial chains, and are defined as follows.

Definition 11. Let δ be an extended simplicial chain, then δ is *normal* if and only if:

$$f_\delta(\mathbf{Q}) = 1, \forall \mathbf{Q} \in FF_\delta. \quad \blacklozenge$$

The next step is, given a free-form solid, to find a normal extended simplicial chain that uniquely determines and represents it. In three dimensions, case of study of this paper, our free-form solids have the boundary composed by a set of algebraic patches of types 3-patch and 1-patch, created in the initial triangulation interpolating process. Therefore, from the boundary of the free-form solid we establish the set of extended cells (tetrahedra and *ffc*'s) and their associated coefficients that compose the normal extended simplicial chain for this solid:

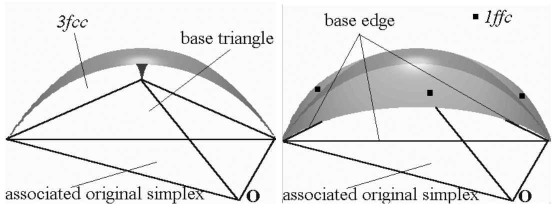
· Each *triangle* t_i, which belongs to initial triangulation and base triangle of any 3-patch, determines an *original simplex* (tetrahedron) formed by the three vertices of t_i and the coordinates origin **O**.

· Each *algebraic patch* (3-patch or 1-patch) determines a *free-form cell* (3*fcc* or 1*fcc*) according to definition 3. Each *ffc* is related to only one original simplex called its *associated original simplex*; for 3*ffc*'s, the associated original simplex is that determined by the base triangle of the 3-patch, which defines the 3*ffc*, and the coordinates origin **O**.

Since a 1-patch (secondary patch) appears in the filling holes process and it is associated with a 3-patch (main patch), then the original simplex associated with a 1*ffc* is the same that is associated with the 3*ffc* whose 3-patch is the main patch of this 1-patch (secondary patch). Figure 8 shows an example of these concepts. In this way, we obtain the extended cells (original simplices and *ffc*'s) to be used in the construction of the normal extended simplicial chain whose associated free-form solid be equal to the constructed from the initial triangulation. If it is necessary the construction of a *double-tetrahedra* [Bajaj95] in order to interpolate a triangle of the initial triangulation, then the two resulting *ffc*'s have, logically, the same associated original simplex. The next step is to assign the coefficient α_i to each extended cell:

· To each *original simplex*, a tetrahedron, we assign as its coefficient α_i the sign of its signed volume.

· To each *ffc*, we assign as its coefficient α_i the value of its sign, calculated according to definitions 4 and 5 (*sign of* 3*ffc* and *sign of* 1*ffc* respectively).



(a) (b)

Figure 8: *(a) Cell* 3*ffc, (b) Cell* 3*ffc and cells* 1*ffc's and associated original simplex.*

We have chosen the coordinates origin **O** as auxiliary vertex to construct the original simplices because of the efficiency in the coefficient calculation, although obviously we can select any other point. Therefore, the equivalence between free-form solids and normal extended simplicial chains is established with the following theorem.

Theorem 1. Let *FF* be a free-form solid with an initial triangulation formed by n triangles $t_1, t_2, ..., t_n$ (triangle vertices are counter clock-wise oriented), m^i algebraic patches have been constructed for triangle t_i to interpolate the initial triangulation; let S_i be the original simplex (tetrahedron) determined by the origin **O** and the triangle t_i whose coefficient is s_i=*sign of* S_i; and let $ffc_{i1}, ffc_{i2}, ..., ffc_{im^i}$ be the *ffc*'s generated to interpolate t_i, whose coefficients are a_{i1}=*sign of* ffc_{i1}, a_{i2}=*sign of* ffc_{i2}, ..., a_{im^i}=*sign of* ffc_{im^i}, then:

$$FF = FF_\delta, \text{ where } \delta = \sum_{i=1}^{n} (s_i S_i + \sum_{j=1}^{m^i} a_{ij} ffc_{ij})$$

Proof. See [Ruiz98b] ♠

Example shown in figure 9 represents the construction of a free-form solid in 2D. In this example is shown the solid, its initial polygon with edges t_i (equivalent to a 3D initial triangulation) and the construction of the solid step by step which is equivalent to evaluate the function $f_\delta(\mathbf{Q})$ for every point in the plane, where dark regions have a negative value for f_δ and clear regions have a positive one. In 2D each t_i has only one *ffc* associated, noted ffc_i. The problems cited in section 2, which require separation half-spaces, are now solved trivially. As example, figure 10 shows both the normal extended simplicial chain for the solid in figure 1 and the result of evaluate $f_\delta(\mathbf{Q})$ step by step. In figure 12 the step by step evaluation of f_δ for a solid in 3D is shown. In this figure, red zones have a negative value and blue zones have a positive one.

BOOLEAN OPERATIONS

Once we have established the correspondence between free form solids and normal extended simplicial chains, the next step is to obtain Boolean operations in terms of operations for extended simplicial chains. In this way, the operations are simplified, because the work is done homogeneously with simple elements, avoiding particular cases. These operations require calculating the intersection of extended cells (original simplices and *ffc*'s). In general, the intersection of extended cells is not an extended cell, but rather a set of points (possibly not connected) which can be decomposed easily in extended cells for the 2D case [Ruiz97], although this decomposition is not so simple in 3D. A possible method for the latter is reconstruct the non-planar part of the intersection by means of techniques as described in [Peters87] obtaining a initial triangulation for the intersection. Therefore, we can associate with the result of regular intersection of two extended cell E and E' a normal extended simplicial chain as we have shown previously. This decomposition in extended cells is represented by means of the function $ExCell(E \cap E')$.

Theorem 2. If FF_1 and FF_2 are two free-form solids, $\delta_1 = \sum_{i=1}^{n} a_i E_i$ and $\delta_2 = \sum_{j=1}^{m} b_j F_j$ are their associated normal extended simplicial chains, then we can express Boolean operations between FF_1 and FF_2 in terms of operations between chains δ_1 and δ_2 as follows:

$$FF_\delta = FF_1 \cap FF_2 : \delta = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i\, b_i)\, ExCell(E_i \cap F_j)$$

$$FF_\delta = FF_1 \cup FF_2 : \delta = \delta_1 + \delta_2 - \delta_{FF_1 \cap FF_2}$$

$$FF_\delta = FF_1 - FF_2 : \ \delta = \delta_1 - \delta_{FF_1 \cap FF_2}$$

Proof. See [Ruiz98a] ♠

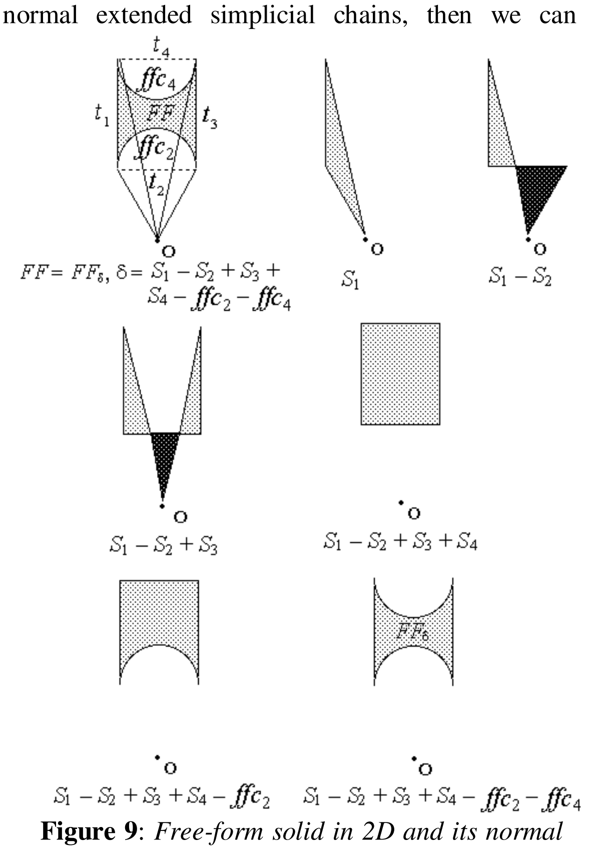


Figure 9: *Free-form solid in 2D and its normal extended simplicial chain.*

Figure 11 shows an example of the intersection of two free-form solids in 2D (we use 2D for the same reasons given for figure 9). In this figure e_i and f_i are the edges as t_i in figures 9 and 10, and E_i and F_i are the original simplices that together with ffc_2 form the normal extended simplicial chains associated with these solids according to theorem 1. The representation of the intersection solid ($FF_\cap$) is done in a constructive manner, taking into account that the pair of extended cells whose intersection is empty are not considered. Figure 13 shows final evaluation of the function f_δ for several extended simplicial chains associated with CSG free-form solids.

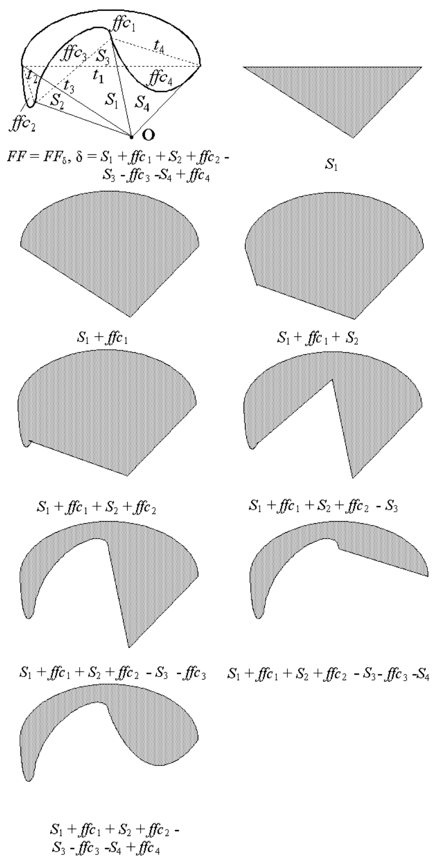


Figure 10: *Solution for solid in figure* 1.

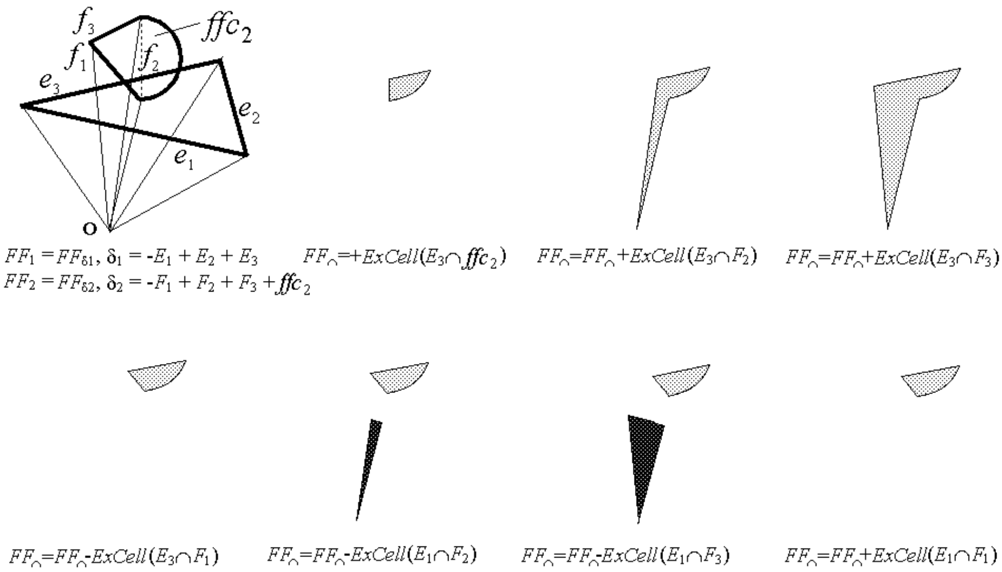


Figure 11: *Free-form solid intersection in 2D.*

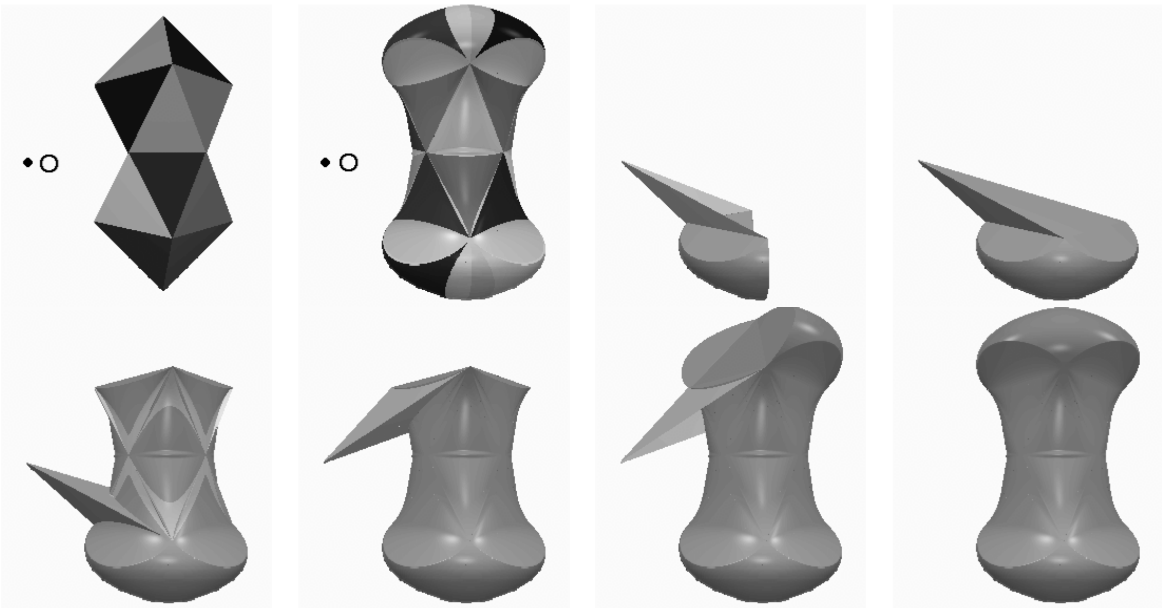


Figure 12: *Initial triangulation, free-form solid in 3D, and its normal extended simplicial chain step by step.*

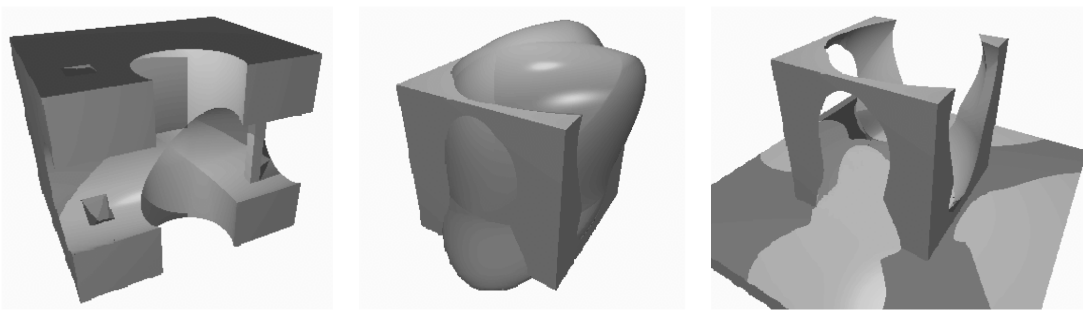


Figure 13: Evaluation of extended simplicial chains for CSG free-form solids.

CONCLUSIONS

Based on the concept of extended simplicial chain, a mathematical model for free-form solid modelling has been presented, particularised for the case of solids whose boundary is defined by a set of algebraic patches. By using it, we can express in a uniform manner both free-form solids and Boolean operations between them. Since the method to define the solid is based on algebraic operations, it avoids both to deal with particular cases and to use strange elements to the model, for example separation half-spaces, due to algebraic operations are accumulative, instead of Boolean operations used in other works. Although in this paper the method has been developed for the case of solids described by algebraic patches, a basic property is the independence from both dimension and surface used to define the solid when it has been decomposed in a set of original simplices and *ffc*'s (the case of 2D with free-form cells defined by conic sections and cubic Bézier curves is already developed [Ruiz97]). Although our decomposition of free-form solids in extended cells could suppose a high number of operations for the algorithms which work with them, really to operate with an extended cell is simple and, since extended cells have common zones, many operations can be simplified. Furthermore, for the case of algebraic patches, the number of extended cells depends on initial triangulation goodness.

Data structures to store this model are analogous to B-rep structures; in fact, B-rep is the starting point for immediate generation of the extended simplicial chain for a free-form solid. In this sense, we can view extended simplicial chains as an intermediate representation scheme that provides simple, homogeneous and robust solutions for fundamental geometric algorithms [Ruiz97, Ruiz99], Boolean operations, representation conversion [Ruiz98a], and many other algorithms, such as collision test, visual operations, ray-reps, ..., can be easily obtained.

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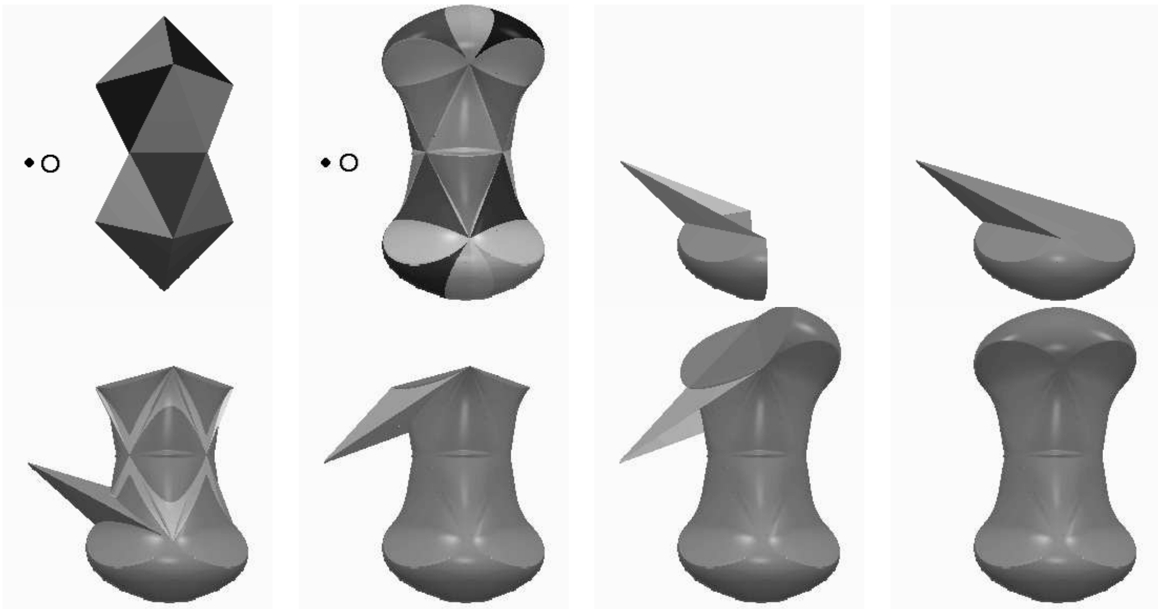

Figure 12: *Initial triangulation, free-form solid in 3D, and its normal extended simplicial chain step by step.*