TRIGONOMETRIC SPLINES AND GEOMETRIC CONTINUITY OF SURFACES

Przemysław Kiciak
Instytut Matematyki Stosowanej i Mechaniki, Uniwersytet Warszawski
ul. Banacha 2, 02–097 Warszawa, Poland, e–mail: przemek@mimuw.edu.pl

ABSTRACT

Trigonometric splines are introduced as a tool for description of a smooth surface in a neighbourhood of a common corner of patches being the surface components. This description, developed for the case of curvature continuity, significantly narrows the gap between the necessary and known sufficient conditions of compatibility of partial derivatives.

Keywords: Geometric continuity of surfaces, curvature continuity, trigonometric splines.

INTRODUCTION

The geometric continuity of a surface of order r (G^r continuity) is defined as the existence of local regular parameterizations, continuous together with partial derivatives up to the order r. In many applications there is a need of constructing G^1 or G^2 continuous surfaces. These properties of surfaces have the geometric interpretation respectively as the continuity of tangent plane and curvature. For special purposes, geometric continuity of even higher order may be required.

Various aspects of geometric continuity between two surface patches are discussed in [DeR90], [Deg90], [Jüt92], [Kah83], [Kic96], [Liu90], [Zhe92], and [Zhe95]. The necessary and sufficient conditions for G^r continuity in this case seem to be fully recognized and practical constructions of smoothly joined pairs of patches are known. However, linear equations that must be satisfied by derivatives of the patches at their common corner (known as the compatibility conditions), are fully solved only in case of G^1 continuity. Only some particular solutions of the equations corresponding to the G^2 case, given in [Du92], and G^r in [Pet92], are known and used in constructions (see e.g. [Ye96]).

Compatibility conditions for G^2 continuity, written in the direct form ([Du92]), involve 14 vectors of partial and mixed partial derivatives per patch, and this is one of the reasons of difficulties in their analysis. It becomes much simpler when the compatibility conditions are written in terms of trigonometric splines. This paper presents such an analysis.

1 PIECEWISE POLYNOMIAL FUNCTIONS OF TWO VARIABLES

1.1 Partitions of the full angle

Let $\Delta = \{\alpha_0, \ldots, \alpha_{k-1}\}$ be a set of k numbers such that $\alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_0 + 2\pi$. Such a set will be called a **partition of the full angle**, or shortly a **partition**. The numbers α_i are the **junction points** of the trigonometric splines defined later.

Throughout this paper we use the indices in a cyclic way (modulo k). Moreover, for $i = k-1$, $[\alpha_i, \alpha_{i+1}]$ denotes the interval $[\alpha_{k-1}, \alpha_0 + 2\pi]$. We use the notation $\delta_i = \alpha_{i+1} - \alpha_i$ ($\delta_{k-1} = \alpha_0 + 2\pi - \alpha_{k-1}$). Obviously $\sum_{i=0}^{k-1} \delta_i = 2\pi$.

We denote $\boldsymbol{u}_i = [\cos\alpha_i, \sin\alpha_i]$, $\boldsymbol{v}_i = [-\sin\alpha_i, \cos\alpha_i]$. If all $\delta_i < \pi$ then the k halflines $\{a\boldsymbol{u}_i : a \geq 0\}$ partition the plane $\mathbb{R}^2$ to k convex cones $C_i = \{a\boldsymbol{u}_i + b\boldsymbol{u}_{i+1} : a, b \geq 0\}$ (if there exists i such that $\delta_i > \pi$ then one of the cones is non–convex). For convenience, the halflines $\{a\boldsymbol{u}_i : a \geq 0\}$ will be called the **junction halflines**. We introduce k systems of coordinates x_i, y_i in $\mathbb{R}^2$, corresponding to the orthonormal bases $\{\boldsymbol{u}_i, \boldsymbol{v}_i\}$; we have $[x, y] = x_i\boldsymbol{u}_i + y_i\boldsymbol{v}_i$.

1.2 Compatibility conditions at a common corner of patches

Let us consider a common corner of k patches forming a G^r–continuous surface. There exists a regular parameterization of this surface of class C^r in the neighbourhood of the corner. Suppose that the common curves of the patches are images of line segments meeting at the point $(0, 0)$, corresponding to the corner. Directions of these line segments determine a partition Δ, and the partition of the plane tangent to the surface at the corner by the halflines tangent to the curves is an affine image of Δ. We are going to take a close look at the scalar functions that describe the coordinates of the surface.

Suppose that in each cone C_i we have a function f_i, continuous together with sufficiently many partial derivatives in a vicinity of $(0, 0)$. If the function $f : \mathbb{R}^2 \to \mathbb{R}$, such that $f|_{C_i} = f_i$ for all i, is C^r continuous, then there occur some dependencies among the Taylor expansions of the functions f_i at the point $(0, 0)$. These dependencies are known as the **compatibility conditions**.

It turns out that the global compatibility conditions involve the terms of order up to $2r$. Any change of a derivative of the function f_i of higher order may require a modification only of the functions in two adjacent cones C_{i-1} and C_{i+1} in order to obtain again a function of class C^r. Therefore we may restrict our attention to functions consisting of polynomial pieces f_i of degree $2r$. Such a function will be called a **piecewise polynomial function of degree** $2r$ **over** Δ.

1.3 Piecewise polynomial functions of two variables in polar coordinates

Any algebraic polynomial is a sum of homogeneous terms. We are going to represent these terms as functions of polar coordinates r, φ in the plane $\mathbb{R}^2$. Using the formulae $x = r\cos\varphi$, $y = r\sin\varphi$, the identity $\cos^2\varphi + \sin^2\varphi = 1$ and the formulae for the sine and cosine of the sum of angles, by induction we can prove (see Remark 1)

Lemma 1 *The vector space of homogeneous algebraic polynomials of two variables of degree n, denoted by $\mathcal{H}_n(\mathbb{R}^2)$, is isomorphic with the space of odd or even, depending on n, trigonometric polynomials of degree n, denoted by $\mathcal{T}_n(S^1)$.*

Remark 1 Odd and even trigonometric polynomials satisfy respectively the conditions $\forall_\alpha f(\alpha + \pi) = -f(\alpha)$ and $\forall_\alpha f(\alpha + \pi) = f(\alpha)$. In particular, for polynomials of degree up to 2 we have

$$\begin{aligned}
&n=0\colon\ r^0=1,\\
&n=1\colon\ r\cos\varphi=x,\ r\sin\varphi=y,\\
&n=2\colon\ r^2=x^2+y^2,\ r^2\cos 2\varphi=x^2-y^2, r^2\sin 2\varphi=2xy.
\end{aligned}$$

Crossing out the factor r^n defines the isomorphism of particular interest to us. Hereafter it will be called the **canonical mapping** of $\mathcal{H}_n(\mathbb{R}^2)$ onto $\mathcal{T}_n(S^1)$. One can also use the bases of $\mathcal{H}_n(\mathbb{R}^2)$ consisting of shifted trigonometric functions, for instance for a fixed $\alpha \in \mathbb{R}$

$$\begin{aligned}
&\{r^n, r^n\cos 2(\varphi-\alpha), r^n\sin 2(\varphi-\alpha), \ldots,\\
&\quad r^n\cos n(\varphi-\alpha), r^n\sin n(\varphi-\alpha)\} \quad \text{for } n \text{ even},\\
&\{r^n\cos(\varphi-\alpha), r^n\sin(\varphi-\alpha), \ldots,\\
&\quad r^n\cos n(\varphi-\alpha), r^n\sin n(\varphi-\alpha)\} \quad \text{for } n \text{ odd}.
\end{aligned}$$

Now, let $f \in C^r(\mathbb{R}^2)$ be a piecewise polynomial function over Δ, described by polynomials f_i of degree not higher than $2r$, as discussed in Section 1.2. It may be represented in the form

$$f(x,y) \;=\; s_0(\varphi) + r s_1(\varphi) + \cdots + r^{2r} s_{2r}(\varphi), \tag{1}$$

where $s_0, \ldots, s_{2r}$ are r times continuously differentiable periodic functions, called hereafter **trigonometric splines**, which consist of pieces being odd or even trigonometric polynomials of degrees respectively $0, \ldots, 2r$, meeting at the junction points $\alpha_0, \ldots, \alpha_{k-1}$. The canonical mapping, that is the restriction to the unit circle, is also an isomorphism of the space of piecewise n–ic homogeneous polynomial functions of class C^r over Δ, denoted by $\mathcal{P}_\Delta^{(n,r)}$, and the corresponding space of trigonometric splines, denoted by $\mathcal{S}_\Delta^{(n,r)}$.

It is easy to notice that the only trigonometric splines of class C^r of degree $0, \ldots, r$ are trigonometric polynomials. The trigonometric splines of higher degrees will be investigated in Section 2.

Let $q(u,v) = p(s(u,v), t(u,v))$. Then $q_v = s_v p_s + t_v p_t$, $q_{vv} = s_{vv} p_s + t_{vv} p_t + s_v^2 p_{ss} + 2 s_v t_v p_{st} + t_v^2 p_{tt}$. After substituting $u = r$, $v = \varphi_i = \varphi - \alpha_i$, $s(u,v) = x_i = r\cos\varphi_i$, $t(u,v) = y_i = r\sin\varphi_i$, we can solve the equations with respect to the partial derivatives of f_i at $(0,0)$:

$$\begin{aligned}
f_i(0,0)&=s_0(\alpha_i) & f_{i\,x_i}(0,0)&=s_1(\alpha_i)\\
f_{i\,y_i}(0,0)&=s_1'(\alpha_i) & f_{i\,x_ix_i}(0,0)&=2\,s_2(\alpha_i)\\
f_{i\,x_iy_i}(0,0)&=s_2'(\alpha_i) & f_{i\,y_iy_i}(0,0)&=2\,s_2(\alpha_i)+s_2''(\alpha_i)\\
f_{i\,x_ix_ix_i}(0,0)&=6\,s_3(\alpha_i) & f_{i\,x_ix_iy_i}(0,0)&=2\,s_3'(\alpha_i)\\
f_{i\,x_iy_iy_i}(0,0)&=3\,s_3(\alpha_i)+s_3''(\alpha_i) & f_{i\,x_ix_ix_ix_i}(0,0)&=24\,s_4(\alpha_i)\\
f_{i\,x_ix_ix_iy_i}(0,0)&=6\,s_4'(\alpha_i) & f_{i\,x_ix_iy_iy_i}(0,0)&=8\,s_4(\alpha_i)+2\,s_4''(\alpha_i)
\end{aligned} \tag{2}$$

Derivatives of f of higher order may be found by considering subsequent derivatives of q.

The relation between the derivatives of the function f and the corresponding trigonometric splines is significant for smooth surface design. Suppose that a designer has modelled a net of curves, which then have to be filled by patches. The derivatives of order n of the curves at their common point determine the corresponding points of a trigonometric spline curve of degree n. If a curve of interpolation to these points exists then its derivatives may in turn be translated to the mixed partial derivatives of the patches under construction. The analysis of Lagrange interpolation problems in the spaces of trigonometric splines, among others, helps to understand the restrictions existing for the shapes of the curves, which are otherwise hard to explain.

2 TRIGONOMETRIC SPLINES

The following theorem is a basis of the further analysis of the compatibility conditions. Its proof is worth attention, as it can help to understand the dependence of possible shapes of surfaces around a common corner on the partition determined by the halflines tangent to the boundary curves of patches.

Theorem 1 *Let Δ be a partition of the full angle, and let m be the number of pairs $\{\alpha_i, \alpha_j\} \subset \Delta$, such that $\alpha_j = \alpha_i + \pi$. The dimension of the space $\mathcal{S}_\Delta^{(n,r)}$ is equal to* $\max\{n+1, k(n-r), m(n-r)+n+1\}$.

Proof: If $n \le r$ then the space $\mathcal{S}_\Delta^{(n,r)}$ is $\mathcal{T}_n(S^1)$, of dimension $n+1$. Let $n > r$. We consider the space $\mathcal{P}_\Delta^{(n,r)}$, which (under the canonical mapping) is isomorphic with $\mathcal{S}_\Delta^{(n,r)}$. If $f_1(x,y)$ and $f_2(x,y)$ are homogeneous polynomials of degree n, meeting with the continuity of derivatives up to the order r along the halfline $\{a\boldsymbol{u_i} : a \ge 0\}$, then their difference is a linear combination of the $n-r$ polynomials $y_i^n, x_i y_i^{n-1}, \ldots, x_i^{n-r-1} y_i^{r+1}$. Let $f_{i,j+}^n = x_i^j y_i^{n-j}$ if $y_i \ge 0$ and 0 otherwise. If $\alpha_i + \pi \in \Delta$ then the functions $f_{i,0+}^n, \ldots, f_{i,n-r-1+}^n$ are elements of $\mathcal{P}_\Delta^{(n,r)}$, which is a direct sum of the space spanned by these functions and the space $\mathcal{P}_{\Delta\setminus\{\alpha_i+\pi\}}^{(n,r)}$. Hence the dimension of $\mathcal{P}_\Delta^{(n,r)}$ (and $\mathcal{S}_\Delta^{(n,r)}$) is $m(n-r)$ plus the dimension of the space $\mathcal{P}_{\Delta_0}^{(n,r)}$, where Δ_0 is a partition obtained by removing from Δ the m numbers $\alpha_i + \pi$.

In the following we deal with partitions $\Delta_0 = \{\alpha_0, \ldots, \alpha_{k-m-1}\}$, containing no pairs $\{\alpha_i, \alpha_i + \pi\}$. To find the dimension of $\mathcal{P}_{\Delta_0}^{(n,r)}$ we transform the partition, by replacing all $\alpha_i > \alpha_0 + \pi$ by $\alpha_i - \pi$. It is easy to notice that the space $\mathcal{P}_{\widetilde{\Delta}_0}^{(n,r)}$, corresponding to the new partition $\widetilde{\Delta}_0 = \{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{k-m-1}\}$, is isomorphic with $\mathcal{P}_{\Delta_0}^{(n,r)}$. To find the dimension of the two spaces, we construct another space V, isomorphic with them.

The $k-m$ halflines, inclined at the angles $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{k-m-1}$, partition the plane $\mathbb{R}^2$ to $k-m$ cones, of which one is non–convex. There exists a line l, crossing all these halflines. The

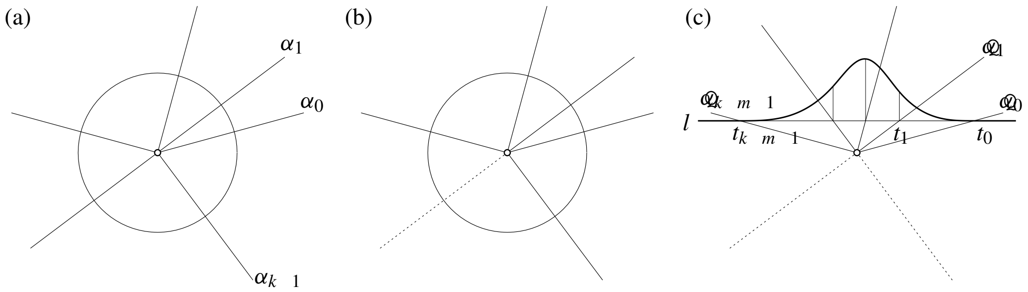

Figure 1: Transformations of the partition in the proof of Theorem 1: a) Δ, b) Δ_0, c) $\widetilde{\Delta}_0$

line l is divided by the halflines into $k - m + 1$ intervals, of which two are infinite. A function $f \in \mathcal{P}^{(n,r)}_{\widetilde{\Delta}_0}$, restricted to this line is a polynomial spline function of degree $\le n$, having r continuous derivatives and described by the same polynomial in both infinite intervals. The space V of such splines is the direct sum of the space of polynomials of degree $\le n$ (of dimension $n + 1$) and of the space of B–spline functions of degree n, over the sequence of $k - m$ knots of multiplicity $n - r$ (its dimension is $\max((k - m)(n - r) - n - 1, 0)$). □

The proof of Theorem 1 allows us to construct a basis of the space of splines corresponding to any given partition. This basis may be modified to obtain bases more convenient in particular applications.

The lemmas that follow, describe some isomorphisms of different spaces of trigonometric splines, which will be used in further investigation. The first of them consists of rather obvious observations, so we omit its proof.

Lemma 2 *Let $a, b, c, d \in \mathbb{R}$, $ad - bc \ne 0$. Let $\Delta = \{\alpha_0, \ldots, \alpha_{k-1}\}$ and $\widetilde{\Delta} = \{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{k-1}\}$ be two partitions such that for all i*

$$\begin{aligned} \cos\tilde{\alpha}_i &= (a\cos\alpha_i + b\sin\alpha_i)/r_i, \\ \sin\tilde{\alpha}_i &= (c\cos\alpha_i + d\sin\alpha_i)/r_i, \end{aligned} \qquad (3)$$

where $r_i = \sqrt{(a^2 + c^2)\cos^2\alpha_i + (ab + cd)\sin 2\alpha_i + (b^2 + d^2)\sin^2\alpha_i}$.

(1) *Spaces $\mathcal{P}^{(n,r)}_{\Delta}$ and $\mathcal{P}^{(n,r)}_{\widetilde{\Delta}}$ of piecewise homogeneous n–ic polynomials of class C^r over the partitions Δ and $\widetilde{\Delta}$ are isomorphic. In particular, the formula $f(x, y) = \tilde{f}(ax + by, cx + dy)$ defines an isomorphism between these spaces.*

(2) *Spaces $\mathcal{S}^{(n,r)}_{\Delta}$ and $\mathcal{S}^{(n,r)}_{\widetilde{\Delta}}$ of trigonometric splines of degree n of class C^r over Δ and $\widetilde{\Delta}$ are isomorphic.*

(3) *If a trigonometric spline of interpolation to $x_0, \ldots, x_{k-1}$ at $\alpha_0, \ldots, \alpha_{k-1}$ exists in $\mathcal{S}^{(n,r)}_{\Delta}$ then a spline of interpolation to $r_0^{-n}x_0, \ldots, r_{k-1}^{-n}x_{k-1}$ at $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{k-1}$ exists in $\mathcal{S}^{(n,r)}_{\widetilde{\Delta}}$. Moreover, the subspaces of $\mathcal{S}^{(n,r)}_{\Delta}$ and $\mathcal{S}^{(n,r)}_{\widetilde{\Delta}}$, whose elements are functions vanishing respectively on Δ and $\widetilde{\Delta}$ are isomorphic.*

Remark 2 Another isomorphism (in fact the identity) is obtained by renumbering a partition, for example $\tilde{\alpha}_0 = \alpha_1, \ \ldots, \ \tilde{\alpha}_{k-2} = \alpha_{k-1}, \ \tilde{\alpha}_{k-1} = \alpha_0 + 2\pi$. We use the renumbering, for instance to obtain $\alpha_2 - \alpha_0 < \pi$, if this is possible and convenient in the proofs of theorems given later, without writing this explicitly.

Lemma 3 *Let l_0, l_1, l_2 and $\tilde{l}_0, \tilde{l}_1, \tilde{l}_2$ be halflines in $\mathbb{R}^2$, having the origin at $(0, 0)$ and inclined respectively at the angles $\alpha_0, \alpha_1, \alpha_2$ and $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2$, such that $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_0 + \pi$ and $\tilde{\alpha}_0 < \tilde{\alpha}_1 < \tilde{\alpha}_2 < \tilde{\alpha}_0 + \pi$. There exists a linear transformation of $\mathbb{R}^2$ mapping l_i on $\tilde{l}_i$ for $i = 0, 1, 2$.*

Proof: Three different halflines l_0, l_2 and l_1 as above may be seen as positive halfaxes of a coordinate system in $\mathbb{R}^2$ and their bisectrix. Hence a linear transformation which maps such a triple onto the other one exists. □

Below we investigate the spaces $\mathcal{S}^{(3,2)}_{\Delta}$ and $\mathcal{S}^{(4,2)}_{\Delta}$, related to the compatibility conditions for curvature continuous surfaces.

2.1 Cubic trigonometric splines

Let $\Delta = \{\alpha_0, \ldots, \alpha_{k-1}\}$ be a partition of the full angle, as described in Section 1.1. We are going to analyze the Lagrange interpolation problem in $\mathcal{S}^{(3,2)}_{\Delta}$. Each element of this space is a function $s(\varphi)$ of class C^2, which in each interval $[\alpha_i, \alpha_{i+1}]$ is given by

$$\begin{aligned} s(\varphi) &= a_i\cos(\varphi - \alpha_i) + b_i\sin(\varphi - \alpha_i) + \\ &\quad + c_i\cos 3(\varphi - \alpha_i) + d_i\sin 3(\varphi - \alpha_i). \end{aligned} \qquad (4)$$

To solve interpolation problems, for each interval $[\alpha_i, \alpha_{i+1}]$ we construct (if possible) the local Hermite basis of the space of odd cubic trigonometric polynomials, which in this case is a set of functions $\{H_{i,00}, H_{i,01}, H_{i,10}, H_{i,11}\}$, such that

$$\begin{bmatrix} H_{i,00}(\alpha_i) & H'_{i,00}(\alpha_i) & H_{i,00}(\alpha_{i+1}) & H'_{i,00}(\alpha_{i+1}) \\ H_{i,01}(\alpha_i) & H'_{i,01}(\alpha_i) & H_{i,01}(\alpha_{i+1}) & H'_{i,01}(\alpha_{i+1}) \\ H_{i,10}(\alpha_i) & H'_{i,10}(\alpha_i) & H_{i,10}(\alpha_{i+1}) & H'_{i,10}(\alpha_{i+1}) \\ H_{i,11}(\alpha_i) & H'_{i,11}(\alpha_i) & H_{i,11}(\alpha_{i+1}) & H'_{i,11}(\alpha_{i+1}) \end{bmatrix}$$

is the identity matrix. Having local Hermite bases for all intervals, with any numbers x_i and y_i one can obtain a function of class C^1, described in $[\alpha_i, \alpha_{i+1}]$ by the formula

$$\begin{aligned} s(\varphi) &= H_{i,00}(\varphi)x_i + H_{i,01}(\varphi)y_i + \\ &\quad H_{i,10}(\varphi)x_{i+1} + H_{i,11}(\varphi)y_{i+1}, \end{aligned} \qquad (5)$$

and $s(\alpha_i) = x_i$ and $s'(\alpha_i) = y_i$. Moreover, one may require the continuity of the second order derivative by imposing the condition $s''(\alpha_i)_- = s''(\alpha_i)_+$ for all i. The result is a system

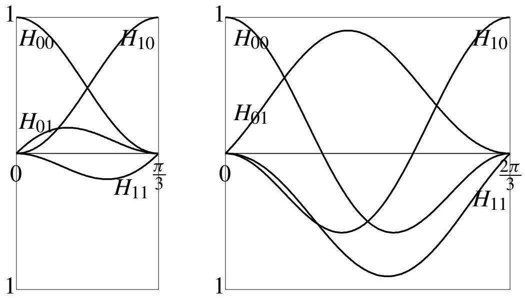

Figure 2: Local Hermite basis functions of the space of odd cubic trigonometric polynomials

of linear equations with unknown y_i, resembling the one obtained in the well known construction of periodic cubic polynomial spline of interpolation.

To find the local Hermite basis we write both representations: (4) and (5) in the form

$$s(\varphi) \;=\; [a_i, b_i, c_i, d_i]T_i(\varphi) \;=\; [x_i, y_i, x_{i+1}, y_{i+1}]H_i(\varphi), \tag{6}$$

where the matrices T_i and H_i represent respectively the trigonometric and Hermite bases:

$$T_i(\varphi) \;=\; \begin{bmatrix} \cos(\varphi-\alpha_i) \\ \sin(\varphi-\alpha_i) \\ \cos 3(\varphi-\alpha_i) \\ \sin 3(\varphi-\alpha_i) \end{bmatrix}, \quad H_i(\varphi) \;=\; \begin{bmatrix} H_{i,00}(\varphi) \\ H_{i,01}(\varphi) \\ H_{i,10}(\varphi) \\ H_{i,11}(\varphi) \end{bmatrix}.$$

By (6) we can write

$$\begin{aligned} [a_i, b_i, c_i, d_i]K_i &= [x_i, y_i, x_{i+1}, y_{i+1}], & (7)\\ K_i^{-1}T_i(\varphi) &= H_i(\varphi). & (8) \end{aligned}$$

These equalities make sense if the matrix K_i, uniquely determined by (7), is regular. We find

$$K_i \;=\; \begin{bmatrix} 1 & 0 & \cos\delta_i & -\sin\delta_i \\ 0 & 1 & \sin\delta_i & \cos\delta_i \\ 1 & 0 & \cos 3\delta_i & -3\sin 3\delta_i \\ 0 & 3 & \sin 3\delta_i & 3\cos 3\delta_i \end{bmatrix},$$

$$K_i^{-1} \;=\; \begin{bmatrix} \frac{3}{4}(1-\frac{\cos^2\delta_i}{\sin^2\delta_i}) & \frac{3}{2}\frac{\cos^3\delta_i}{\sin^3\delta_i} & \frac{1}{4}(1+\frac{3\cos^2\delta_i}{\sin^2\delta_i}) & -\frac{\cos^3\delta_i}{2\sin^3\delta_i} \\ -\frac{\cos\delta_i}{2\sin\delta_i} & \frac{3}{4\sin^2\delta_i}-\frac{1}{2} & \frac{\cos\delta_i}{2\sin\delta_i} & \frac{1}{2}-\frac{1}{4\sin^2\delta_i} \\ \frac{3\cos\delta_i}{4\sin^2\delta_i} & \frac{9\sin^2\delta_i-6}{4\sin^3\delta_i} & -\frac{3\cos\delta_i}{4\sin^2\delta_i} & \frac{2-3\sin^2\delta_i}{4\sin^3\delta_i} \\ -\frac{1}{4\sin\delta_i} & \frac{3\cos\delta_i}{4\sin^2\delta_i} & \frac{1}{4\sin\delta_i} & -\frac{\cos\delta_i}{4\sin^2\delta_i} \end{bmatrix}.$$

The matrix K_i^{-1} exists when $\delta_i = \alpha_{i+1} - \alpha_i \neq l\pi$, $l \in \mathbb{Z}$, hence the Hermite basis exists for each segment $[\alpha_i, \alpha_{i+1}]$ if $\delta_i \neq \pi$. If the matrix K_i^{-1} is known then the values of the second order derivatives of the Hermite basis functions at α_i may be found by differentiating (8). We can calculate

$$H_i''(\alpha_i) = \begin{bmatrix} -3-6\frac{\cos^2\delta_i}{\sin^2\delta_i} \\ -4\frac{\cos\delta_i}{\sin^2\delta_i} \\ 6\frac{\cos\delta_i}{\sin^2\delta_i} \\ -\frac{2}{\sin\delta_i} \end{bmatrix}, \quad H_{i-1}''(\alpha_i) = \begin{bmatrix} 6\frac{\cos\delta_{i-1}}{\sin^2\delta_{i-1}} \\ \frac{2}{\sin\delta_{i-1}} \\ -3-6\frac{\cos^2\delta_{i-1}}{\sin^2\delta_{i-1}} \\ 4\frac{\cos\delta_{i-1}}{\sin^2\delta_{i-1}} \end{bmatrix}.$$

After substituting this to the equation $s''(\alpha_i)_- = s''(\alpha_i)_+$, we obtain

Theorem 2 *If $\delta_{i-1} = \alpha_i - \alpha_{i-1} \neq \pi$ and $\delta_i = \alpha_{i+1} - \alpha_i \neq \pi$ then the continuity of the second order derivative of a cubic trigonometric spline s at α_i is described by the equation*

$$\begin{aligned} &\sin\delta_i\; y_{i-1} + 2\sin(\delta_{i-1}+\delta_i)\; y_i + \sin\delta_{i-1}\; y_{i+1} \;=\; \\ &3\Big(-\frac{\cos\delta_{i-1}\sin\delta_i}{\sin\delta_{i-1}}x_{i-1} + \frac{\cos 2\delta_{i-1}-\cos 2\delta_i}{2\sin\delta_{i-1}\sin\delta_i}x_i + \frac{\cos\delta_i\sin\delta_{i-1}}{\sin\delta_i}x_{i+1}\Big), \end{aligned} \tag{9}$$

where $x_i = s(\alpha_i)$, $y_i = s'(\alpha_i)$.

Hereafter in this section we assume $\delta_i \neq \pi$ for all i.

As in the case of periodic cubic polynomial splines, Equations (9) form a system of k linear equations with cyclic three–diagonal matrices A_3, B_3

$$A_3Y_3 \;=\; B_3X_3, \tag{10}$$

where

$$\begin{aligned} X_3 &= [x_0, \ldots, x_{k-1}]^T \;=\; [s(\alpha_0), \ldots, s(\alpha_{k-1})]^T, \\ Y_3 &= [y_0, \ldots, y_{k-1}]^T \;=\; [s'(\alpha_0), \ldots, s'(\alpha_{k-1})]^T, \\ A_3 &= \end{aligned}$$

$$\begin{bmatrix} 2\sin(\delta_{k-1}+\delta_0) & \sin\delta_{k-1} & & \sin\delta_0 \\ \sin\delta_1 & 2\sin(\delta_0+\delta_1) & \ddots & \\ & \ddots & \ddots & \sin\delta_{k-3} \\ \sin\delta_{k-2} & & \sin\delta_{k-1} & 2\sin(\delta_{k-2}+\delta_{k-1}) \end{bmatrix},$$

$$B_3 =$$

$$3\begin{bmatrix} \frac{\cos 2\delta_{k-1}-\cos 2\delta_0}{2\sin\delta_{k-1}\sin\delta_0} & \frac{\cos\delta_0\sin\delta_{k-1}}{\sin\delta_0} & & \frac{-\cos\delta_{k-1}\sin\delta_0}{\sin\delta_{k-1}} \\ \frac{-\cos\delta_0\sin\delta_1}{\sin\delta_0} & \frac{\cos 2\delta_0-\cos 2\delta_1}{2\sin\delta_0\sin\delta_1} & \ddots & \\ & \ddots & \ddots & \frac{\cos\delta_{k-2}\sin\delta_{k-3}}{\sin\delta_{k-2}} \\ \frac{\cos\delta_{k-1}\sin\delta_{k-2}}{\sin\delta_{k-1}} & & \frac{-\cos\delta_{k-2}\sin\delta_{k-1}}{\sin\delta_{k-2}} & \frac{\cos 2\delta_{k-2}-\cos 2\delta_{k-1}}{2\sin\delta_{k-2}\sin\delta_{k-1}} \end{bmatrix}.$$

The matrix A_3 is not always regular, hence the periodic trigonometric spline of interpolation to given $x_0, \ldots, x_{k-1}$ not always exists. Moreover, the matrix B_3 may also be singular and therefore the existence of solutions Y_3 for all vectors X_3 is not equivalent to their uniqueness. To discuss the existence and uniqueness of splines of interpolation we will consider also the homogeneous system

$$A_3Y \;=\; 0. \tag{11}$$

A piecewise cubic homogeneous polynomial function f which equals 0 on all junction halflines is called here a **piecewise cubic monkey saddle**. If f is a piecewise cubic monkey saddle and $f(r\cos\varphi, r\sin\varphi) = r^3 s(\varphi)$, then the trigonometric spline s corresponds to a solution of the homogeneous system (11). When the interpolation with trigonometric splines is considered, (10) is a system of k equations with k unknowns y_i. The absence of nonzero monkey saddles over Δ is therefore a necessary condition for the uniqueness and a sufficient condition for the existence of solutions.

Proposition 1 *If $k = 3$ then for any partition $\alpha_0, \alpha_1, \alpha_2$ and any given vector X_3, the system of equations* (10) *has solutions.*

Proof: The following trigonometric polynomials interpolate to the given x_0, x_1, x_2 at $\alpha_0, \alpha_1, \alpha_2$

$$s(\varphi) = \sum_{i=0}^{2} x_i \frac{\sin(\varphi-\alpha_{i-1}) \sin^2(\varphi-\alpha_{i+1})}{\sin \delta_{i-1} \sin^2 \delta_i} + a \prod_{i=0}^{2} \sin(\varphi - \alpha_i),$$

for any $a \in \mathbb{R}$. The above formula describes all solutions of the interpolation problem, because $k = 3$, $\delta_i \neq \pi \;\Rightarrow\; m = 0$, and by Theorem 1 the dimension of $\mathcal{S}_\Delta^{(3,2)}$ is equal to 4. □

Theorem 3 *Let $k = 4$.*

(1) *If $\alpha_2 \neq \alpha_0 + \pi$ and $\alpha_3 \neq \alpha_1 + \pi$ then Equations* (10) *have a unique solution for any X_3.*

(2) *If only one of the equalities $\alpha_2 = \alpha_0 + \pi$ or $\alpha_3 = \alpha_1 + \pi$ holds then the space of solutions of the homogeneous system* (11) *is two–dimensional. Solutions of* (10) *in case of $\alpha_2 = \alpha_0 + \pi$, $\alpha_3 \neq \alpha_1 + \pi$ exist iff $x_0 = -x_2$ and in case of $\alpha_3 = \alpha_1 + \pi$, $\alpha_2 \neq \alpha_0 + \pi$ if $x_1 = -x_3$.*

(3) *If $\alpha_2 = \alpha_0 + \pi$ and $\alpha_3 = \alpha_1 + \pi$ then the space of solutions of* (11) *is two–dimensional and solutions of* (10) *exist for any given vector X_3.*

Proof: **(1)** The assumption $\delta_i \neq \pi$ and $\alpha_{i+2} \neq \alpha_i + \pi$ for all i implies that $\alpha_i \neq \alpha_j \pm l\pi$, for all integer i, j, l ($i \neq j$). Therefore the trigonometric polynomial

$$s(\varphi) \;\;=\;\; \sum_{i=0}^{3} x_i \prod_{\substack{j=0\\ j\neq i}}^{3} \frac{\sin(\varphi-\alpha_j)}{\sin(\alpha_i-\alpha_j)}$$

exists and interpolates to the given values x_i. The uniqueness of the solution follows from Theorem 1, as the dimension of $\mathcal{S}_\Delta^{(3,2)}$ in this case is equal to 4.

(2) If $\alpha_2 = \alpha_0 + \pi$ then Equations (10) take the form

$$\begin{aligned}
E_0&: 2dy_0+s_2y_1 +s_0y_3 = 3\Big(\tfrac{de}{s_2s_0}x_0+\tfrac{c_0s_2}{s_0}x_1 +\tfrac{c_2s_0}{s_2}x_3\Big),\\
E_1&: s_0y_0 +s_0y_2 = 3(-c_0x_0 - c_0x_2),\\
E_2&: s_2y_1-2dy_2+s_0y_3 = 3\Big(\tfrac{c_0s_2}{s_0}x_1-\tfrac{de}{s_0s_2}x_2+\tfrac{c_2s_0}{s_2}x_3\Big),\\
E_3&: s_2y_0 +s_2y_2 = 3(-c_2x_0 - c_2x_2),
\end{aligned}$$

where $s_i = \sin\delta_i$, $c_i = \cos\delta_i$, $d = \sin(\delta_0 + \delta_3)$, $e = \sin(\delta_0 - \delta_3)$. The conditions for the existence of solutions are homogeneous equations with unknown x_i, obtained by taking linear combinations of the above equations with linearly independent vectors of coefficients, so as to eliminate y_i. We have $\mathrm{rank}A_3 = 2$, hence we need two such equations (not necessarily independent if $\mathrm{rank}B_3 < 4$), for instance

$$s_2E_1 - s_0E_3, \quad \text{and} \quad E_0 - \tfrac{2d}{s_0}E_1 - E_2.$$

If $\delta_0 \neq \delta_2$ then both above equations yield the same condition $x_0 + x_2 = 0$.

(3) If $\delta_0 = \delta_2$, that is $\alpha_3 = \alpha_1 + \pi$, then both equations obtained in the proof of **(2)** become $0 = 0$. □

Theorem 4 *Let $k = 5$.*

(1) *If $\alpha_{i+2} \neq \alpha_i + \pi$ for all i then the system of equations* (10) *has a unique solution.*

(2) *If there exists exactly one i such that $\alpha_{i+2} = \alpha_i + \pi$, then the space of solutions of* (11) *is one–dimensional and solutions of* (10) *exist iff $x_{i+2} = -x_i$.*

(3) *If $\alpha_{i+2} = \alpha_i + \pi$ and $\alpha_{i+3} = \alpha_{i+1} + \pi$ for some i then the space of solutions of* (11) *is one–dimensional and solutions of* (10) *exist for any given vector X_3.*

Proof: **(1)** By Lemmas 2 and 3, we can transform the numbers $\alpha_0, \ldots, \alpha_4$ to a convenient particular case. It is possible to obtain in this way the partition $\{0, \alpha_1, \alpha_2, \frac{4\pi}{3}, \frac{5\pi}{3}\}$, thus reducing the number of independent variables to 2. In this case it turns out that

$$\det A_3 = 9\sqrt{3}\, \sin(\alpha_1 - \tfrac{\pi}{3})\, \sin(\alpha_1 - \tfrac{2\pi}{3})\, \sin\alpha_2\, \sin(\alpha_2 + \tfrac{\pi}{3}).$$

In $\{(\alpha_1, \alpha_2) : 0 < \alpha_1 < \alpha_2 < \frac{4\pi}{3}\}$ the only cases when $\det A_3$ vanishes are $\alpha_1 = \frac{\pi}{3}$, $\alpha_1 = \frac{2\pi}{3}$, $\alpha_2 = \frac{2\pi}{3}$, $\alpha_2 = \pi$, and these cases contradict the assumption $\alpha_{i+2} \neq \alpha_i + \pi$ for all i.

(2) The existence of one i such that $\alpha_{i+2} = \alpha_i + \pi$ is equivalent to the existence of four different lines containing the five junction halflines. The dimension of $\mathcal{S}_\Delta^{(3,2)}$ is, by Theorem 1, equal to 5, and there exists a basis consisting of four linearly independent cubic trigonometric polynomials and of the function $f^3_{i,0+}$ defined in the proof of Theorem 1. By Theorem 3, a trigonometric polynomial of interpolation to $x_i, x_{i+1}, x_{i+3}, x_{i+4}$ at $\alpha_i, \alpha_{i+1}, \alpha_{i+3}, \alpha_{i+4}$ exists. However, for all the basis functions we have $f(\alpha_{i+2}) = -f(\alpha_i)$, which implies the necessary and sufficient condition $x_{i+2} = -x_i$ for the existence of solutions of (10).

(3) If there exists i such that $\alpha_{i+2} = \alpha_i + \pi$ and $\alpha_{i+3} = \alpha_{i+1} + \pi$ then there are three lines in $\mathbb{R}^2$ containing the junction halflines. By Theorem 1, $\dim\mathcal{S}_\Delta^{(3,2)} = 6$ and $\mathcal{S}_\Delta^{(3,2)}$ is spanned by four linearly independent cubic trigonometric polynomials and the functions $f^3_{i,0+}$ and $f^3_{i+1,0+}$. Since $f^3_{i,0+}(\alpha_j) \neq 0$ iff $j = i + 1$ and $f^3_{i+1,0+}(\alpha_j) \neq 0$ iff $j = i + 2$ and the problem of interpolation to x_i, x_{i+3}, x_{i+4} at $\alpha_i, \alpha_{i+3}, \alpha_{i+4}$ with cubic trigonometric polynomials has (an infinite set of) solutions, it is clear that there exist splines of interpolation to any given numbers $x_0, \ldots, x_4$. □

The discussion of the existence and uniqueness of solutions of (10) in case of $k \geq 6$ turns out to be more difficult. It may be done by finding the set of zeros of the determinant of the matrix A_3, seen as a function of $\delta_0, \ldots, \delta_{k-2}$, whose domain is the simplex $\{(\delta_0, \ldots, \delta_{k-2}) : \forall_i \delta_i > 0,\ \sum_{i=0}^{k-2} \delta_i < 2\pi\}$. By Lemmas 2 and 3 we can fix $\delta_{k-2} = \delta_{k-1} = \frac{\pi}{3}$ and investigate the set of zeros of $\det A_3(\delta_0, \ldots, \delta_{k-4})$ in the simplex $\{(\delta_0, \ldots, \delta_{k-4}) : \forall_i \delta_i > 0,\ \sum_{i=0}^{k-4} \delta_i < \frac{4\pi}{3}\}$. Figure 3 shows the set of zeros of $\det A_3$ for $k = 6$. This set is a surface of a rather complicated shape and what we know about the existence and uniqueness of solutions in the case $k = 6$ (important in surface design) is summarized in

Proposition 2 *Let $k = 6$.*

(1) *If $\alpha_{i+3} = \alpha_i + \pi$ for $i = 0, 1, 2$ then a solution of* (10) *exists for all vectors X_3; the space of solutions of the homogeneous system* (11) *is one–dimensional.*

(2) *In other cases the dimension of the $\mathcal{S}_\Delta^{(3,2)}$ is equal to 6. If the junction halflines lie on four different lines in $\mathbb{R}^2$ then the matrix A_3 is regular.*

(3) *If the junction halflines lie on five or six lines and* $\det A_3 = 0$ *then there exists a linear equation, that must be satisfied by X_3 to ensure the existence of solutions of* (10).

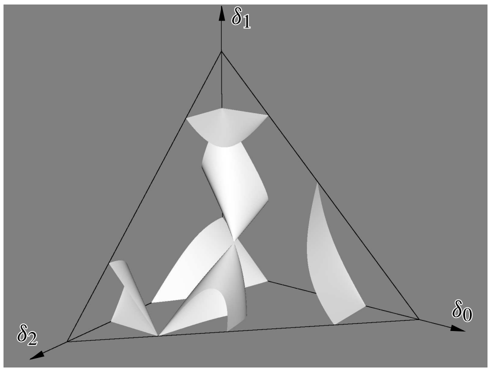

Figure 3: Set $\det A_3 = 0$ for $k = 6$, $\delta_4 = \delta_5 = \frac{\pi}{3}$.

Proof: (1) By Theorem 1, the dimension of the space $\mathcal{S}_\Delta^{(3,2)}$ is 7. We have

$$\mathcal{S}_\Delta^{(3,2)} = \operatorname{span}\{f_{0,0+}^3, \ldots, f_{4,0+}^3, \sin^2(\varphi-\alpha_0)\sin(\varphi-\alpha_1), \sin(\varphi-\alpha_0)\sin(\varphi-\alpha_1)\sin(\varphi-\alpha_2)\}.$$

The Vandermonde matrix $[f_j(\alpha_i)]_{ij}$ of the first six functions given above is nonsingular, hence these functions allow us to solve the interpolation problem for an arbitrary vector X_3. The last function vanishes at all points α_i.

(2) If the junction halflines lie on four different lines, then by Lemmas 2 and 3, the problem of regularity of the matrix A_3 may be reduced to one of the following three cases:

- $\Delta = \{0, \alpha_1, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}\}$, $0 < \alpha_1 < \frac{\pi}{2}$: we find $\det A_3 = \frac{9}{2}(\sin 2\alpha_1 - 1)$. If $\alpha_1 \neq \frac{\pi}{4}$ then $\det A_3 \neq 0$.
- $\Delta = \{0, \alpha_1, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}\}$, $0 < \alpha_1 < \frac{\pi}{2}$: we obtain $\det A_3 = -\frac{9}{2}(1 + \sin 2\alpha_1) \neq 0$.
- $\Delta = \{0, \alpha_1, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, $0 < \alpha_1 < \frac{\pi}{4}$: we have $\det A_3 = -\frac{9}{2}\cos^2\alpha_1 \neq 0$.

(3) If the partition Δ contains at most one pair $\{\alpha_i, \alpha_i + \pi\}$, then by removing α_i we obtain a partition Δ_0 such that its 5 junction halflines lie on 5 different lines. By Theorem 4, all interpolation problems in $\mathcal{S}_{\Delta_0}^{(3,2)}$ have unique solutions. The space $\mathcal{S}_\Delta^{(3,2)}$ is the direct sum of $\mathcal{S}_{\Delta_0}^{(3,2)}$ and span$\{s\}$, where s corresponds to a nonzero solution of (11) with $\det A_3 = 0$. As $s(\alpha_i) = 0$, the number x_i must be equal to the value of the spline of interpolation in $\mathcal{S}_{\Delta_0}^{(3,2)}$ at α_i. $\square$

By Theorem 1 and due to $m \leq \lfloor\frac{k}{2}\rfloor$, if $k \geq 7$ then the dimension of the space $\mathcal{S}_\Delta^{(3,2)}$ is equal to k. Numerical experiments for $k = 7, 8$ detected the existence of partitions such that $\det A_3 = 0$ (see Figure 4). We have, however, the following theorem, which indicates some partitions (with at least 7 elements) for which all interpolation problems have unique solutions and is significant for the numerical solving of the system (10).

Theorem 5 *If $\delta_{i-1} + \delta_i < \frac{2\pi}{3}$ for all i, then the matrix A_3 is diagonally dominant (and hence regular).*

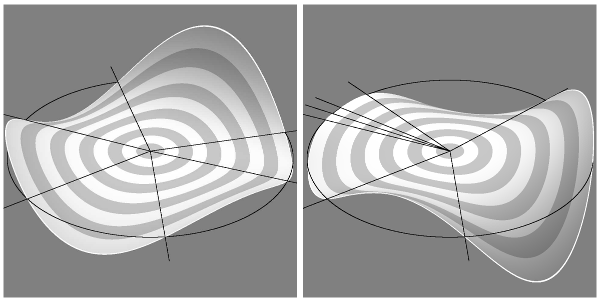

Figure 4: Examples of piecewise cubic monkey saddles with 6 and 7 junction halflines

Proof: Suppose that $\delta_{i-1} + \delta_i = \alpha_{i+1} - \alpha_{i-1} < \pi$. Then $\sin\delta_{i-1} > 0$, $\sin\delta_i > 0$, $\sin(\delta_{i-1} + \delta_i) > 0$. Let

$$\begin{aligned} f(\delta_{i-1}, \delta_i) &= |2\sin(\delta_{i-1}+\delta_i)| - |\sin\delta_{i-1}| - |\sin\delta_i| \\ &= 2\sin(\delta_{i-1}+\delta_i) - \sin\delta_{i-1} - \sin\delta_i \\ &= 2\sin\tfrac{x}{2}(2\cos\tfrac{x}{2} - \cos\tfrac{y}{2}), \end{aligned}$$

where $x = \delta_{i-1} + \delta_i$, $y = \delta_{i-1} - \delta_i$. For $0 < \delta_{i-1} + \delta_i = x < \frac{2\pi}{3}$ we have $2\cos\frac{x}{2} > 2\cos\frac{\pi}{3} = 1 \geq \cos\frac{y}{2}$ and $\sin\frac{x}{2} > 0$, hence $f(\delta_{i-1}, \delta_i) > 0$. If this holds for all i, then the matrix A_3 is diagonally dominant. $\square$

2.2 Quartic trigonometric splines

Let $s(\varphi)$ be a function such that for $\varphi \in [\alpha_i, \alpha_{i+1}]$ we have

$$\begin{aligned} s(\varphi) &= a_i + b_i\cos 2(\varphi-\alpha_i) + c_i\sin 2(\varphi-\alpha_i) + \\ &\quad d_i\cos 4(\varphi-\alpha_i) + e_i\sin 4(\varphi-\alpha_i). \end{aligned} \tag{12}$$

The function $r^4 s(\varphi)$ is a piecewise quartic homogeneous polynomial function of the variables $x = r\cos\varphi$, $y = r\sin\varphi$.

Note that since $m \leq \lfloor\frac{k}{2}\rfloor$, if $k \geq 5$ then $\dim\mathcal{S}_\Delta^{(4,2)} = 2k$ (see Theorem 1). As in the case of trigonometric polynomials of degree 3, for each interval $[\alpha_i, \alpha_{i+1}]$ we can describe a change of local bases using a nonsingular matrix K_i:

$$[a_i, b_i, c_i, d_i, e_i]K_i = [x_i, y_i, z_i, x_{i+1}, y_{i+1}], \tag{13}$$

$$K_i^{-1}T_i(\varphi) = H_i(\varphi), \tag{14}$$

where

$$T_i(\varphi) = \begin{bmatrix} 1 \\ \cos 2(\varphi-\alpha_i) \\ \sin 2(\varphi-\alpha_i) \\ \cos 4(\varphi-\alpha_i) \\ \sin 4(\varphi-\alpha_i) \end{bmatrix}, \quad H_i(\varphi) = \begin{bmatrix} H_{i,00}(\varphi) \\ H_{i,01}(\varphi) \\ H_{i,02}(\varphi) \\ H_{i,10}(\varphi) \\ H_{i,11}(\varphi) \end{bmatrix}.$$

To shorten the notation we omit below the index i of the local basis functions H. After some laborious calculations we obtain

Lemma 4 *The functions H_{00}, H_{01}, H_{02}, H_{10}, H_{11}, defined by (14) and by the matrices*

$$K_i = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 4(1+\cos 2\delta_i) & \cos 2\delta_i & -2\sin 2\delta_i \\ 0 & 2 & 4\sin 2\delta_i & \sin 2\delta_i & 2\cos 2\delta_i \\ 1 & 0 & 8(3+\cos 2\delta_i)\cos 2\delta_i & \cos 4\delta_i & -4\sin 4\delta_i \\ 0 & 4 & 8(3+\cos 2\delta_i)\sin 2\delta_i & \sin 4\delta_i & 4\cos 4\delta_i \end{bmatrix},$$

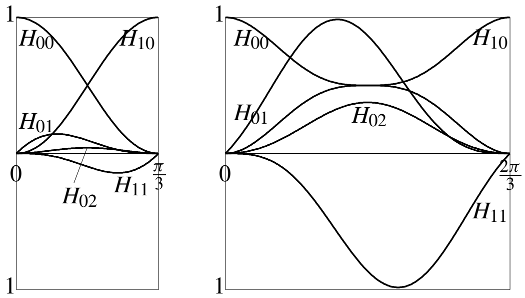

Figure 5: Local basis functions of the space of even quartic trigonometric polynomials

$$K_i^{-1} \;=$$
$$\begin{bmatrix} \frac{1}{2} & \frac{\cos 2\delta_i}{\cos 2\delta_i - 1} & \frac{\sin 2\delta_i \cos 2\delta_i}{(\cos 2\delta_i - 1)^2} & \frac{1+\cos 2\delta_i}{2(1-\cos 2\delta_i)} & -\frac{\sin 2\delta_i \cos 2\delta_i}{2(\cos 2\delta_i - 1)^2} \\ -\frac{3}{4}\frac{\sin 2\delta_i}{(\cos 2\delta_i - 1)^2} & \frac{(\cos 2\delta_i + 1)\sin 2\delta_i}{2(\cos 2\delta_i - 1)^2} & \frac{\cos 2\delta_i + 2}{2(1-\cos 2\delta_i)} & \frac{(1-2\cos 2\delta_i)\sin 2\delta_i}{4(\cos 2\delta_i - 1)^2} & \frac{2\cos 2\delta_i + 1}{4(\cos 2\delta_i - 1)} \\ \frac{\cos 2\delta_i + 2}{8(1-\cos 2\delta_i)} & \frac{\cos 2\delta_i + 1}{4(\cos 2\delta_i - 1)} & \frac{\sin 2\delta_i}{4(\cos 2\delta_i - 1)} & \frac{\cos 2\delta_i}{8(1-\cos 2\delta_i)} & \frac{\sin 2\delta_i}{8(1-\cos 2\delta_i)} \\ \frac{1}{2} & \frac{\cos 2\delta_i}{1-\cos 2\delta_i} & -\frac{\cos 2\delta_i \sin 2\delta_i}{(\cos 2\delta_i - 1)^2} & \frac{\cos 2\delta_i + 1}{2(\cos 2\delta_i - 1)} & \frac{\cos 2\delta_i \sin 2\delta_i}{2(\cos 2\delta_i - 1)^2} \\ \frac{3}{4}\frac{\sin 2\delta_i}{(\cos 2\delta_i - 1)^2} & -\frac{\sin 2\delta_i}{(\cos 2\delta_i - 1)^2} & \frac{1}{2(\cos 2\delta_i - 1)} & \frac{\sin 2\delta_i}{4(\cos 2\delta_i - 1)^2} & \frac{1}{4(1-\cos 2\delta_i)} \end{bmatrix}$$

are linearly independent and satisfy the following equations:

$$H_{00}(\alpha_i)=H'_{01}(\alpha_i)=H_{10}(\alpha_{i+1})=H'_{11}(\alpha_{i+1})=H''_{02}(\alpha_i)=H''_{02}(\alpha_{i+1})=1,$$
$$H'_{00}(\alpha_i)=H_{01}(\alpha_i)=H_{02}(\alpha_i)=H'_{02}(\alpha_i)=H_{10}(\alpha_i)=H'_{10}(\alpha_i)=H_{11}(\alpha_i)=$$
$$H'_{11}(\alpha_i)=H_{00}(\alpha_{i+1})=H'_{00}(\alpha_{i+1})=H_{01}(\alpha_{i+1})=H'_{01}(\alpha_{i+1})=H_{02}(\alpha_{i+1})=$$
$$H'_{02}(\alpha_{i+1})=H'_{10}(\alpha_{i+1})=H_{11}(\alpha_{i+1})=H''_{01}(\alpha_{i+1})=H''_{11}(\alpha_i)=0,$$
$$H_{00}(\varphi)+H_{10}(\varphi)=1,$$
$$H_{00}(\alpha_i+\varphi)=H_{10}(\alpha_{i+1}-\varphi),$$
$$H_{01}(\alpha_i+\varphi)=-H_{11}(\alpha_{i+1}-\varphi),$$
$$H_{02}(\alpha_i+\varphi)=H_{02}(\alpha_{i+1}-\varphi),$$
$$H''_{00}(\alpha_i)=-H''_{10}(\alpha_i)=-H''_{00}(\alpha_{i+1})=H''_{10}(\alpha_{i+1})=\frac{4(\cos 2\delta_i+2)}{\cos 2\delta_i-1},$$
$$H''_{01}(\alpha_i)=-H''_{11}(\alpha_{i+1})=\frac{6\sin 2\delta_i}{\cos 2\delta_i-1}.$$

Theorem 6 *If $\delta_{i-1} = \alpha_i - \alpha_{i-1} \neq \pi$ and $\delta_i = \alpha_{i+1} - \alpha_i \neq \pi$ then the continuity of the second order derivative of the function s at α_i is described by the equation*

$$\begin{aligned} &(\cos 2\delta_{i-1} - 1)(\cos 2\delta_i - 1)(z_i - z_{i-1}) + \\ &\quad 6(\sin 2(\delta_{i-1} + \delta_i) - \sin 2\delta_{i-1} - \sin 2\delta_i)\, y_i \;=\; \\ &-4(\cos 2\delta_{i-1} + 2)(\cos 2\delta_i - 1)(x_i - x_{i-1}) + \\ &\quad +4(\cos 2\delta_i + 2)(\cos 2\delta_{i-1} - 1)(x_{i+1} - x_i). \end{aligned} \qquad (15)$$

We have $x_i = s(\alpha_i)$ and $y_i = s'(\alpha_i)$.

Proof: By Lemma 4, the set of functions $\{H_{00}, H_{01}, H_{10}, H_{11}\}$ is the Hermite basis of the space spanned by these functions. The sides of the equation

$$\begin{aligned} &-\frac{4(\cos 2\delta_{i-1}+2)}{\cos 2\delta_{i-1}-1} x_{i-1} + z_{i-1} + \frac{4(\cos 2\delta_{i-1}+2)}{\cos 2\delta_{i-1}-1} x_i - \frac{6\sin 2\delta_{i-1}}{\cos 2\delta_{i-1}-1} y_i \\ &= \frac{4(\cos 2\delta_i+2)}{\cos 2\delta_i-1} x_i + \frac{6\sin 2\delta_i}{\cos 2\delta_i-1} y_i + z_i - \frac{4(\cos 2\delta_i+2)}{\cos 2\delta_i-1} x_{i+1}. \end{aligned} \quad (16)$$

are expressions for the second order derivatives of trigonometric polynomials meeting at α_i. From this equation, after some calculation, we obtain (15). □

We assume $\delta_i \neq \pi$ for all i. Equations (15) may be written in the matrix form

$$A_4 Z_4 + B_4 Y_4 \;=\; C_4 X_4, \qquad (17)$$

where

$$\begin{aligned} X_4 &= [x_0, \ldots, x_{k-1}]^T \;=\; [s(\alpha_0), \ldots, s(\alpha_{k-1})]^T, \\ Y_4 &= [y_0, \ldots, y_{k-1}]^T \;=\; [s'(\alpha_0), \ldots, s'(\alpha_{k-1})]^T, \\ Z_4 &= [z_0, \ldots, z_{k-1}]^T, \\ A_4 &= \operatorname{diag}\big((\cos 2\delta_{k-1} - 1)(\cos 2\delta_0 - 1), \ldots, \\ &\qquad (\cos 2\delta_{k-2} - 1)(\cos 2\delta_{k-1} - 1)\big) \cdot D_-, \\ B_4 &= \operatorname{diag}\big(6(\sin 2(\delta_{k-1} + \delta_0) - \sin 2\delta_{k-1} - \sin 2\delta_0), \ldots, \\ &\qquad 6(\sin 2(\delta_{k-2} + \delta_{k-1}) - \sin 2\delta_{k-2} - \sin 2\delta_{k-1})\big), \\ C_4 &= \operatorname{diag}\big(-4(\cos 2\delta_{k-1} + 2)(\cos 2\delta_0 - 1), \ldots, \\ &\qquad -4(\cos 2\delta_{k-2} + 2)(\cos 2\delta_{k-1} - 1)\big) \cdot D_- + \\ &\quad \operatorname{diag}\big(4(\cos 2\delta_0 + 2)(\cos 2\delta_{k-1} - 1), \ldots, \\ &\qquad 4(\cos 2\delta_{k-1} + 2)(\cos 2\delta_{k-2} - 1)\big) \cdot D_+, \end{aligned}$$

$$D_- = \begin{bmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}, \qquad D_+ \;=\; \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ 1 & & & -1 \end{bmatrix}.$$

Theorem 7 **(1)** *If $\delta_i + \delta_{i+1} \neq \pi$ (that is $\alpha_{i+2} \neq \alpha_i + \pi$) for all i, then the system of equations* (17) *with fixed X_4 and Z_4 has a unique solution Y_4.*

(2) *The homogeneous system $A_4 Z = 0$ has solutions of the form $Z = a[1, \ldots, 1]^T$, $a \in \mathbb{R}$. The system* (17) *with X_4 and Y_4 fixed has solutions Z_4 if and only if the following equation is satisfied*

$$\begin{aligned} &\textstyle\sum_{i=0}^{k-1} \frac{\sin 2(\delta_{i-1}+\delta_i) - \sin 2\delta_{i-1} - \sin 2\delta_i}{(\cos 2\delta_{i-1}-1)(\cos 2\delta_i-1)}\, y_i \;=\; \\ &\qquad \textstyle\sum_{i=0}^{k-1} \frac{4(\cos 2\delta_{i-1} - \cos 2\delta_i)}{(\cos 2\delta_{i-1}-1)(\cos 2\delta_i-1)}\, x_i. \end{aligned} \qquad (18)$$

(3) *If $k = 4$ and $\alpha_2 = \alpha_0 + \pi$, $\alpha_3 = \alpha_1 + \pi$, then the system of equations* (17) *with X_4 and Y_4 fixed has solutions Z_4.*

(4) *In all other cases, the system of equations* (17) *with unknown vectors Y_4 and Z_4 has solutions and the space of solutions of the homogeneous system $A_4 Z + B_4 Y = 0$ is k–dimensional.*

Proof: **(1)** Let $f(\delta_{i-1}, \delta_i) = \sin 2(\delta_{i-1} + \delta_i) - \sin 2\delta_{i-1} - \sin 2\delta_i$. We have $f(\delta_{i-1}, \delta_i) = 2\sin x(\cos x - \cos y)$, where $x = \delta_{i-1} + \delta_i$, $y = \delta_{i-1} - \delta_i$. Hence in the set $\{(\delta_{i-1}, \delta_i) : \delta_{i-1} > 0, \delta_i > 0, \delta_{i-1} + \delta_i < 2\pi, \delta_{i-1} \neq \pi, \delta_i \neq \pi\}$ the function f is equal to 0 iff $\delta_{i-1} + \delta_i = \pi$. If $\delta_i + \delta_{i+1} \neq \pi$, for all i then all the elements on the diagonal of B_4 are nonzero and thus B_4 is a nonsingular matrix.

(2) We have rankA_4 = rankD_- = $k - 1$ and the systems of equations $A_4 Z = 0$ and $D_- Z = 0$ are equivalent. Equation (18) is obtained by adding and rearranging the equations (16).

(3) If $k = 4$ and $\alpha_2 = \alpha_0 + \pi$, $\alpha_3 = \alpha_1 + \pi$, then $B_4 = 0$. Equation (18), a necessary and sufficient condition for the existence of Z_4 satisfying (17) with given X_4 and Y_4, becomes $0 = 0$.

(4) It is obvious that the partitions $\alpha_0 < \ldots < \alpha_{k-1} < \alpha_0 + 2\pi$ such that for all i $\alpha_{i+2} = \alpha_i + \pi$ may occur only for $k = 4$. In all other cases (also when $k = 4$) the matrix B_4 is nonzero and the rank of the $k \times 2k$ matrix $[A_4, B_4]$ is equal to k. The system of equations with this matrix is therefore consistent and the dimension of the space of solutions of the homogeneous system is equal to k. $\square$

3 CONCLUSION

Although the analysis of trigonometric splines for the case of curvature continuity involves a considerable amount of laborious calculation, and so far is incomplete for more than 5 curves meeting at a common corner, it makes a significant progress in understanding of the compatibility conditions. In fact, the analysis presented above, revealing shape parameters available with a given partition, seems to cover all the cases useful in practical design of surfaces. A construction of a curvature continuous surface might look as follows: after establishing for each corner a partition, one can construct four trigonometric spline curves of degree $1, \ldots, 4$. They determine a system of compatible partial and mixed partial derivatives of patches, which may then be individually constructed and reparameterized.

The cubic trigonometric splines, corresponding to any surface that may be obtained with use of the construction given in [Ye96] (using the "standard" approach), are cubic trigonometric polynomials. We have shown that in many cases this approach makes use of all the possible shape parameters, however with the trigonometric splines we can indicate the cases where more parameters are available.

To investigate the compatibility conditions for the geometric continuity of order r higher than 2, one may analyze the appropriate Lagrange interpolation problems in the spaces of trigonometric splines of class C^r. This would be even more laborious than in the case of G^2 continuity, however the approach using trigonometric splines seems easier than any other one, and it is the complexity of the subject that makes the thing difficult, regardless of the method of analysis.

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