Boolean representation of general planar polygons

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ABSTRACT

There are many applications of the geometric modelling in which it is necessary to accomplish decompositions of the geometric objects that intervene in the model, to facilitate the analysis and design of the algorithms implicated in the resolution of particular problems. Triangulations, trapezoidations, convex decompositions and BSP (binary space partitioning trees) are some of the methods used for the polygon representation. In this work we present a new type of decomposition, based on triangles, valid for closed polygons, manifold and non-manifold, with or without holes. The structure as, well as the necessary algorithms for the calculation of the representation, is easy to implement. The resulting formula can be seen as a particular case of CSG representation of polygons, in which the primitives are triangles.

Keywords: polygons, Boolean expresions, decomposition of polygons, triangles, CSG

INTRODUCTION

In many of the applications of the geometric modelling we need to decompose the geometric objects that intervene in the representation of the model. So it facilitates the analysis and design of the algorithms implicated in the resolution of particular problems. Triangulations, trapezoidations, convex decompositions and binary space partitioning trees, are some of the methods used for the polygon representation. In [Du95], [Dobki88], [Prepa88], [O'Rou94] and [Thiba87] some of the decomposition methods are presented and bibliography of them is given. In this work we present a new model for the closed polygon decomposition (with or without holes, manifold and non-manifold).The structure, as well as the necessary algorithms for the calculation of the representation, is easy to implement. The accomplished decomposition is very suitable for the calculation of some properties of the polygons, such as the area, as well as for the resolution of test inclusion and ray-tracing algorithms. The resulting formula can be seen as a particular case of CSG representation of polygons in which the primitives are triangles.

Taking into account that a B-rep model of polyhedrons is determined by its boundary and that this is formed by polygons, the decomposition that we present would also be applied to modelling in 3D, being one of the fields in which we are working currently [Feito98a].

The presented structure avoids the calculation and processing of other geometric elements (triangles, trapezoids or convex parts) as well as the need to calculate. The triangles that intervene are determined directly by the B-rep of the polygon.

In this work we present the process in which the polygon decomposition is based and the algorithm for the obtainment of such decompositions is detailed.

For the obtainment of the decomposition we consider the B-rep of the polygon and we suppose that the minimal bounding box (MBBOX) of the polygon is calculated. Throughout all the work we will suppose that the origin of coordinates coincides with the left inferior corner of the MBBOX.

FUNDAMENTAL DEFINITIONS

We will give in this section the definitions that will permit us to establish a relationship among triangles that have a vertex in common. We will suppose that the polygon is determined by its vertices and edges, sorted in counter-clockwise order (Fig.1.a). The edges that determine holes have an opposite direction: they are in clockwise order (Fig.1.b). This implies that the interior

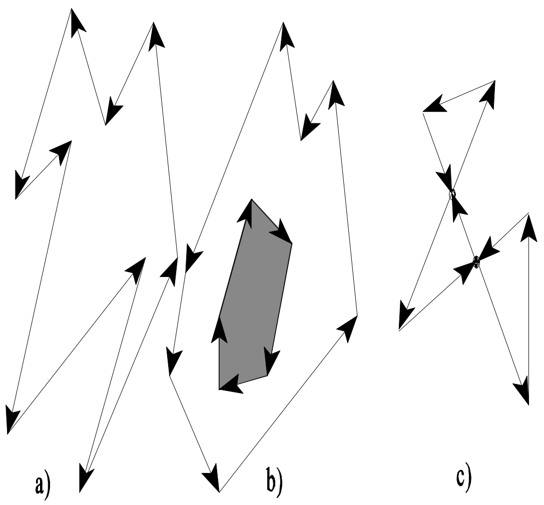


Figure 1

of the solid always remains to the left. The representation is valid also for polygons with self-intersections, though in this case it is considered that the intersection point is also a vertex: we will consider it a non-manifold closed polygon (see Fig.1.c).

In the case of non-manifold closed polygons it is supposed that we have a list of individual shells, being each of these shells a simple polygon.

Definition 1. Let *t be* a triangle of vertices $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$. We will say that it is *original* if one of them coincides with the origin of coordinates. The edges that contain the origin will be designated *original* edges and the others *non original.*

Definition 2. Given an original triangle $t = \mathbf{OP}_1\mathbf{P}_2$, its sign, sign(t), is defined as:

$$sign(t) = \begin{cases} +1 \;\; if \;\; x_1 y_2 - x_2 y_1 > 0 \\ -1 \;\; if \;\; x_1 y_2 - x_2 y_1 < 0 \\ \;\;0 \;\; if \;\; x_1 y_2 - x_2 y_1 = 0 \end{cases} \qquad \textbf{(1)}$$

being **O** the origin of coordinates and $\mathbf{P}_1=(x_1,y_1)$, $\mathbf{P}_2=(x_2,y_2)$ the cartesian coordinates of the other two vertices.

It is evident that every original triangle can have two signs, depending on the order of the two vertices different from the origin of coordinates. If $\mathbf{O}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are in counterclockwise order, the sign of the triangle is +1 and - 1 in clockwise. If the three points are in the same line, we have a special triangle whose sign is 0.

To calculate the decomposition, we will work with original triangles determined by the origin and the edges of the polygon. Because of this we can suppose throughout the work that the non original edges intersect each other, at least, in one of its vertices. In the case of non-manifold polygons we will suppose that every intersection determines a new vertex, and so, new edges and new associated triangles (see Fig.1.c.)

In all cases, it is simple to demonstrate (see [Feito97]) that the intersections of semilines that depart from the origin, with edges that form positive triangles, in different points of the vertices, imply a transition from inside to outside the polygon and the intersections with negative triangles imply a transition from outside to inside the polygon.

In order to establish a hierarchy among original triangles associated with a polygon, we introduce the next definition:

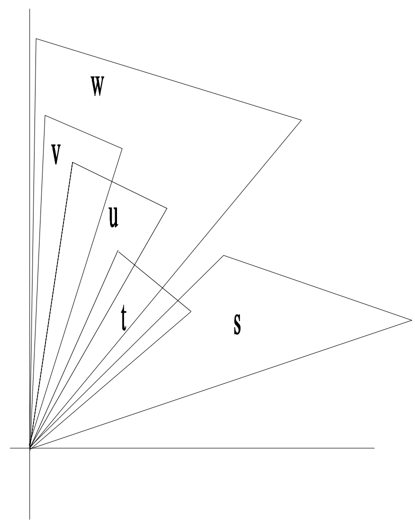


Figure 2

Definition 3. (Subordinated triangles). Let $t=\mathbf{OP_1P_2}$ and $s=\mathbf{OQ_1Q_2}$ be two original triangles. We will say that *t* is subordinated to *s*, $t \lhd s$, if, and only if, $t=s$ or, if $t \neq s$, if it exists a point belonging to the unidimensional interior of the segment $\mathbf{P_1P_2}$ that also belongs to the interior of the triangle *s*. That is to say:

$t \lhd v \Leftrightarrow (t=s)$ or (if $t \neq s, \exists \lambda \in (0,1) \,|\, \lambda \cdot \mathbf{P_1} + (1-\lambda) \cdot \mathbf{P_2} \in \mathrm{int}(s)$)

In Fig.2 some subordinated triangles are shown. The triangle $\mathbf{t \lhd u}$; $\mathbf{u \lhd v}$; $\mathbf{t \lhd w}$; $\mathbf{t \lhd s}$; $\mathbf{v \lhd w}$

Our aim is to obtain the decomposition of a polygon in original triangles, the left inferior vertex of MBBOX being the origin.

We can study the subordination of two original triangles t=$\mathbf{OP_1P_2}$ and s=$\mathbf{OQ_1Q_2}$ evaluating the relative position of the vertices $\mathbf{P}_1$ and $\mathbf{P}_2$ of t with respect to s. We have three possibilities:

· $\mathbf{P}_1$ and $\mathbf{P}_2$ are both within s. In this case there is subordination of t to s.

· Only one of the points $\mathbf{P}_1$ and $\mathbf{P}_2$ is in t. In this case there is also subordination of t to s.

· $\mathbf{P}_1$ and $\mathbf{P}_2$ are both outside s. In this case there is subordination of t to s if the segment $\mathbf{P}_1\mathbf{P}_2$ intersects the original edges of s.

Using the previous tests it is very easy to determine if one triangle is subordinated to another. Note that a point $\mathbf{Q}$ belongs to a triangle $\mathbf{OP}_1\mathbf{P}_2$ if the triangles $\mathbf{QP}_1\mathbf{P}_2$, $\mathbf{QP}_2\mathbf{O}$ and $\mathbf{QOP}_1$ have the same sign [Feito97].

Now, we will give a lemma to be used later.

Lemma 1. Let t and s be two original triangles whose non original edges are intersected each other at least in a vertex. If $\mathbf{t} \lhd s$, every semiline, starting in the origin and cutting the non original edges of the two triangles in points different from the vertices, first intersets t and then s.

Proof:

Let r be a semiline that starts in the origin and intersects the non original edges of t and s in $\mathbf{Q}$ and $\mathbf{Q'}$, respectively. If $t \lhd s$, the point $\mathbf{Q}$ will be necessarily interior to s and so Q will be closer to the origin than Q' (Fig. 3). ■

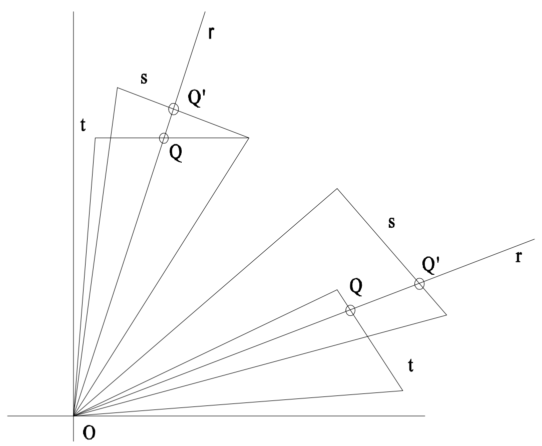


Figure 3

The relation of subordination introduced in the definition 3 is reflexive (by definition) and antisymmetric (consequence of the previous lemma), though it is not transitive as can be easily verified: the triangles t, u and v of figure 2 verify that $t \lhd u$, $u \lhd v$ and however t is not subordinated to v.

We are going to generalize the previous definition to get a relation that fulfils the transitive property in order to obtain an order relation. It will be made by means of the next definition.

Definition 4. Let P={t_i | i=1,..,n } be a set of subordinated original triangles whose non original edges intersect each other at least in a vertex. We will say that t_j is smaller or equal to t_k ($t_j \le t_k$) if a subset of P, P'= {s_1, s_2, ..., s_m} exists with $s_1 = t_j$, $s_m = t_k$, and being $s_l \lhd s_{l+1}$, $\forall$ l, 1≤l≤m-1.

We are going to prove that relation fulfils the reflexive, transitive and antisymmetric properties, and so it will be an order relation (a partial order)..

Theorem 1. In a set of original triangles P={t_i | i=1,..,n }, whose non original edges intersect each other at least in a vertex, the relation introduced in the definition 4 is an order relation.

Proof:

It will be necessary to prove the reflexive, antisymmetric and transitive properties.

First, we are going to demonstrate that the relation is reflexive: Let t_j be a triangle of P. Since every triangle is subordinated to itself, the subset P'={t_j} verifies the definition 4, therefore $t_j \le t_j$,and the relation is reflexive.

Now, we will prove the transitive property. We suppose that $t_i \le t_j$ and that $t_j \le t_k$. For the first relation it exists a subset P'={t_i=s_1,...,t_j=s_m} and for the second relation it exists a subset P''={t_j=u_1,..,t_k=u_q}. The subset P'''={s_1,..,s_m,u_1,...,t_k} clearly verifies the conditions demanded in the definition, therefore $t_i \le t_k$.

Finally, we will prove the antisymmetric property. We suppose that there are two triangles s and t of P, such that $s \le t$ and $t \le$ s. This implies that two subsets of P, P'={s=s_1,...,t=s_m} and P''={t=t_1,..,s=t_q} exist, such that:

$s = s_1 \lhd ... \lhd s_m = t$ and $t = t_1 \lhd ... \lhd t_q = s$

So the subset P'''={s=s_1,...,s_m,t_1,...,t_q=s} determines a cycle for s.

Taking into account that a cycle exists, it is necessary to have an approximation of the non original edges of triangles of P''' toward the origin. As the triangles must be maintained within the MBBOX of P, it is necessary to have a slope change in two consecutive triangles of P''', u_h and u_{h+1}, from positive to negative, if the cycle is on the left ((2) in Fig.4) or from negative to positive if the cycle is on the right ((1) in Fig.4), so some of them will not be subordinated to any other triangle, and this contradicts the relation determined in P'''. ■

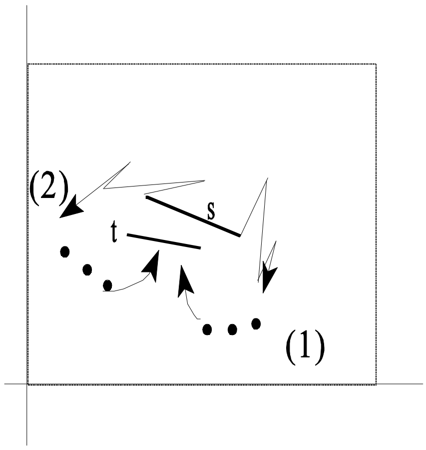


Figure 4

In Fig.5. we can observe an example in which the antisymmetric property is not verified, if we choose as the origin of coordinates the gravity center of the polygon shown. Every triangle is subordinated to the next, forming a cycle. In this case, the defined relation is not an order relation. Because of this, we decompose the polygons, selecting the left inferior vertex of the MBBOX as origin of coordintates, though other singular points can be found as origin of coordinates. For example, we could choose any exterior point to the MBBOX.

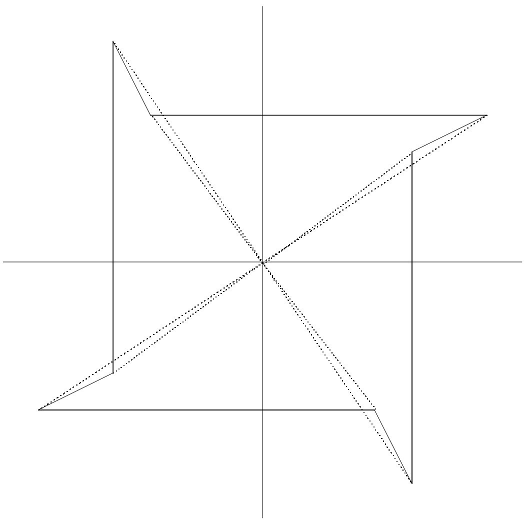


Figure 5

In the next section we are going to apply the results indicated above to obtain the polygon representation.

POLYGON REPRESENTATION

In this section we will see how the previous definitions can be applied in the polygon representation.

The fundamental property to be applied is the following: from the previous order relation we can associate a structure of network ([Feito98b]) to each polygon. We introduce the next definitions:

Definition 5. (index). Let P be a polygon and $\{t_i \mid i=1,..,n\}$ the associated set of original triangles. We define the index of t_i (ind(t_i)) as the depth level in the network determined by the ordered set of triangles of P, expressed from bigger to smaller, and supposing that the maximal elements have value 1 (if an element is minimal and maximal at the same time, as it occurs with the triangles that are not subordinated to any other, these are of index 1).

Definition 6. (cap). Let P be a polygon and $\{t_i \mid i=1,...,n\}$ the associated set of original triangles. We define the cap k of the polygon as the set of original triangles of index k.

By means of these definitions, we will prove that every polygon can be defined as a set of boolean operations accomplished by caps.

First of all we will demonstrate that, according to the previous definitions, all the triangles with index 1, or odd index, have positive sign and those with even index have negative sign.

Lemma 2. Let P be a polygon and $\{t_i \mid i=1,..,n\}$ the associated set of original triangles. Then, if the index of t_i is odd, sign(t_i)>0 and if indice of t_i is even, sign(t_i)<0.

Proof:

By induction.

Let t_i be a triangle of index 1. Thus t_i is notsubordinated to any other original triangle and so, every semiline, that starts from the origin and intersects the non original edge of t_i, will not intersect another edge of P later. This implies that the intersection of this semiline with the non original edge of t_i indicates a transition from inside to outside P, and so t_i is positive.

We suppose that t_j is of index k (k>1). So it will exist an original triangle t_i, with index k-1, such as $t_j \triangleleft t_i$; every semiline starting from the origin and intersecting the non original edges of t_j and t_i, first intersects t_j and then to t_i,without any intersection with other edges between them; so, the intersection of the semiline with t_j implies a transition from outside to inside and *so* t_j is negative.

Similarly, if t_i is negative, t_j must be positive. ■

The next theorem permits us to establish, for any polygon, a boolean expression of original triangles, each one of them appearing only once. For this reason, we are going to build the solid from its caps, from outside toward the origin, using regularized boolean operations [Requi78].

Theorem 2. Let P be a polygon of edges $\{e_1, e_2, .., e_n\}$ and $\{t_i \mid i=1,..,n\}$ the associated set of original triangles. Then, the polygon P can be expressed as:

$$P=(((C_1 -^* C_2) \cup^* C_3)-^*C_4)..\cup^* C_{2*m-1})-^* C_{2*m}))) \quad \textbf{(2)}$$

Where $C_i = \cup \{t_j \mid ind(t_j)=i\}$, and $2{\cdot}m$ the greater index.

Proof:

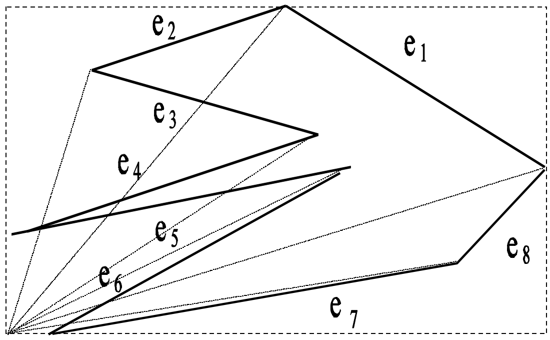
We will prove the double inclusion between the sets determined by the member of eq. 2. Being regularized operations, it will be enough to prove the inclusions only for the points of the interior of each one of the sets.

Let **Q** be a point of the interior of P. If we trace a semiline starting from the origin, passing by **Q** and directed to the infinite, it will exist a first triangle t_j that will be intersected after **Q.** Being **Q** a point interior to P, t_j will be positive and therefore of odd index k. So, the point Q will be in the positive triangle t_j and not in any of its subordinated triangles; therefore Q will be in the cap C_k, with k odd, and not belong to any superior cap of even index, and therefore it will be in the set determined by the second member.

If **Q** belongs to the set indicated in the second member of eq. 2, then there will be an odd value k, such as **Q** is in the cap k, C_k, andnot in another superior cap. So, it must exist a positive triangle t_i of index k such as **Q** belongs to t_i and not to any negative triangle of superior index. Therefore **Q** is in P. ■

Being regularized operations, it is not necessary for the degenerated triangles (determined by colineal edges with the origin) to intervene in the formula. To generate this, it is enough to apply the theorem: first we calculate the possible subordinations and then the caps. Fig.6 shows a decomposition obtained by implementing the proposed decomposition theorem, applied to a simple polygon.

The process of calculating subordinations is simple, therefore it is the calculation of the associated representation.



$$C_1 = t_1 \cup t_2 \cup t_7 \cup t_8 \ ; \ C_2 = t_3 \ ; C_3 = t_4$$

$$C_4 = t_5 \quad ; \quad C_5 = t_6$$

$$P=((((C_1 - C_2)\cup C_3) - C_4)\cup C_5)$$

Figure 6

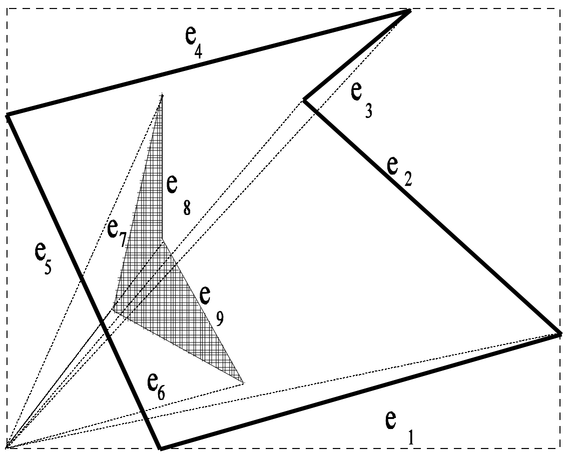
It is evident, from the method followed to obtain the formula, that each positive triangle appears only once. Since the number of positive triangles is finite, the process is finite and the formula is obtained in finite time.

The complexity to calculate all the subordinations is $O(n^2)$, where *n* is the number of vertex of the polygon P: in the worst case, it is neccesary to study the relation among each pair of triangles.

By means of the technique called “sweep line” [Prepa88], applied circularly, it is posible to reduce the complexity of the caps calculation to O(nlogn) ([Feito98b]).

All the development is also valid for closed polygons with holes and for non manifold polygons. Properties encluding this type of polygons has not been used in the previous demonstrations.

In the case of polygons with holes, the edges of their boundary are defined in clockwise order. The subordination corresponding to original triangles, associated with exterior boundary and the boundary of holes, must be studied, obtaining a boolean expression that integrates each triangle in its caps, such as we can observe in Fig.7.



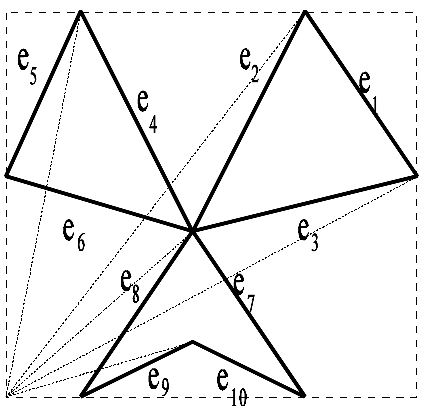
$$C_1 = t_1 \cup t_4 \ ; \ C_2 = t_3 \cup t_8$$

$$C_3 = t_2 \ ; \ C_4 = t_9 \ ; \ C_5 = t_7 \cup t_6 \ ; \ C_6 = t_5$$

$$P = (((((C_1 - C_2) \cup C_3) - C_4) \cup C_5) - C_6)$$

Figure 7

Fig.8. shows a descomposition obtained by implementing the decomposition theorem, applied to a non-manifold closed polygon.



$$C_1 = t_1 \cup t_5 \ ; \ C_2 = t_2 \cup t_3 \ ; \ C_3 = t_4 \cup t_7$$

$$C_4 = t_6 \cup t_{10} \ ; \ C_5 = t_9 \ ; \ C_6 = t_8$$

$$P = (((((C_1 - C_2) \cup C_3) - C_4) \cup C_5) - C_6)$$

Figure 8

CONCLUSIONS AND FUTURE WORKS

A new boolean representation for polygons is presented. The model is valid for general polygons: with or without holes, manifold and nonmanifold. The method for the polygon decomposition avoids the calculation of new geometric elements as well as intersection calculation. The advantage of this method is the simplicity of the algorithms that intervene, as well as of the elements that form the decomposition, and the possibility of expressing any general polygon as a boolean expression of simple primitives..

The obtained formula can be seen as a particular case of CSG representation of polygons, in which the primitive used are triangles.

Future works will be oriented to enhance the preprocess (subordination and index calculation), searching more adequate data structure, and to the dinamization of the decomposition process. Also, we are working in the extension of the model to high dimensions.

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