On a Fundamental Physical Principle Underlying the Point Location Algorithm in Computer Graphics

Sumit Ghosh
Department of Computer Science & Engineering
Arizona State University, Tempe, AZ 85287
sumit.ghosh@asu.edu

Abstract

The issue of point location is an important problem in computer graphics and the study of efficient data structures and fast algorithms is an important research area for both computer graphics and computational geometry disciplines. When filling the interior region of a planar polygon in computer graphics, it is necessary to identify all points that lie within the interior region and those that are outside. Sutherland and Hodgman are credited for designing the first algorithm to solve the problem. Their approach utilizes vector construction and vector cross products, and forms the basis of the "odd parity" rule. To verify whether a test point is within or outside a given planar polygon, a ray from the test point is drawn extending to infinity in any direction without intersecting a vertex. If the ray intersects the polygon outline an odd number of times, the region is considered interior. Otherwise, the point is outside the region. In 3 dimensional space, Lee and Preparata propose an algorithm but their approach is limited to point location relative to convex polyhedrons with vertices in 3D-space. Although it is rich on optimal data structures to reduce the storage requirement and efficient algorithms for fast execution, a proof of correctness of the algorithm, applied to the general problem of point location relative to any arbitrary surface in 3D-space, is absent in the literature. This paper argues that the electromagnetic field theory and Gauss's Law constitute a fundamental basis for the "odd parity" rule and shows that the "odd parity" rule may be correctly extended to point location relative to any arbitrary closed surface in 3D-space.

Keywords: Point location, computer graphics, odd parity rule, Gauss' Law, electromagnetic field theory.

1. Introduction

In computer graphics [1][2][3][13], to determine whether a point lies within or outside a polygon, a ray is drawn starting at the point and extending to infinity in any direction but not intersecting any vertex. If the ray intersects the outline of the polygon an odd number of times, the test point is considered to be within the polygon. Otherwise, the point is outside the polygon. The technique is referred to as the "odd parity" rule. The basic scheme is due to Sutherland and Hodgman [4]. A key element – function "INSIDE" [1], determines whether a point, P, is to the left or right of a boundary, represented by the directed line segment from P_1 to P_2. First, the cross product of $\vec{P_1P_2}$ and $\vec{P_1P}$ is computed. Second, where the cross product is along the positive z-axis, the point P is to the left and thus outside. If it is along the negative z-axis, the point is to the right or inside. In Figure 1, P_3 is considered inside since the cross product of $\vec{P_1P_2}$ and $\vec{P_1P_3}$ is along the negative z-axis. Point P_4 is viewed as outside since the cross product of $\vec{P_1P_2}$ and $\vec{P_1P_4}$ is along the positive z-axis.

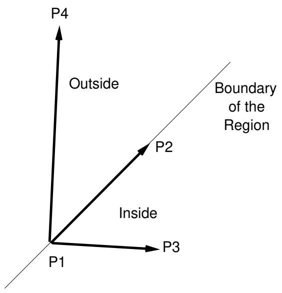

Figure 1: The Basic Sutherland and Hodgman Scheme

The scan-line approach, used in polygon filling, is an extension of the basic approach described earlier. Given a point and a closed polygon, one draws a line through the point extending to infinity. If the line intersects the polygon an odd number of times, assuming that the line does not intersect at any vertex, the point is considered to lie inside the polygon. Otherwise, it lies outside the polygon. Figure 2 shows a complex polygon with two holes in it and a number of points P_1 through P_4, some located within while others are located outside the polygon. The algorithm correctly determines that the points P_2 and P_4 lie inside the polygon since the lines P_2Q_2 and P_4Q_4 intersect the polygon an odd number of times, while the points P_1 and P_3 lie outside the polygon since the lines P_1Q_1 and P_3Q_3 intersect the polygon an even number of times.

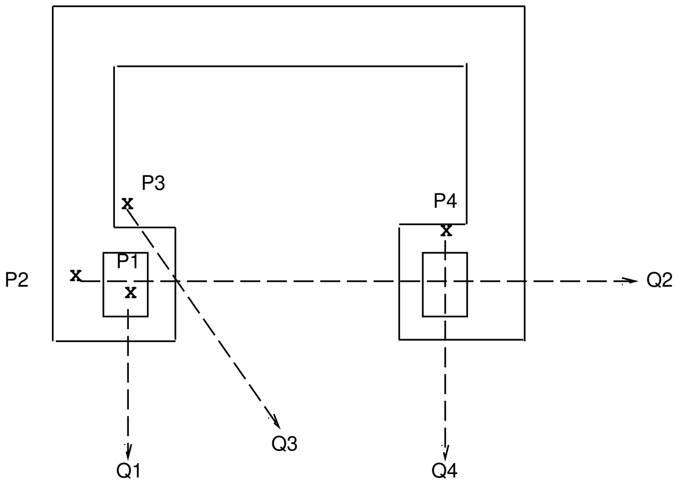

Figure 2: Planar Point Location Utilizing the "Odd Parity" Rule

In the discipline of computational geometry, the problem of point location in a planar subdivision [5] reduces to determining the region of the subdivision occupied by the point. The basic approach is to triangulate any subdivision or a polygon with holes and determine the triangle in which the point lies. The computational geometry literature is rich on techniques – choice of data structures and algorithms, to speed up the result [6][7][8]. Given a subdivision S with n vertices, the key performance measures for the point location algorithm (i) the time for preprocessing, (ii) the space required to store the data structure, and (iii) the time required to search the data structure to locate the point. Lee and Preparata [6] present an approach to determine whether a test point P is located within a convex polyhedron in 3D-space. They select a vertex with the largest Z coordinate and then drop a stereographic projection of the vertex of the polyhedron – a planar graph S', onto a x-y plane. The test point is also projected on the same x-y plane as P'. Next, they apply their planar point location technique to the projected point and planar graph on the x-y plane and argue that if P' is within S', the original point in 3D-space is contained within the convex polyhedron. Edelsbrunner, Guibas, and Stolfi [8] present a space-optimal approach for point location and claim that its efficiency renders it a candidate for rectangular point location in higher dimensions.

The literature in both computer graphics [9][10] and computational geometry [6][[7] focus on efficient data structures and algorithms for fast region filling and point location. However, the literature does not address the underlying principle of the odd parity rule and is unable to argue why the technique works, whether it is guaranteed to work for every planar polygon, and whether it may be extended to any arbitrary polygon in 3D-space.

This paper presents a fundamental physical principle from which the "odd parity" rule may be derived for 3D-space. Thus, given any arbitrary closed surface in 3D-space, the location of any point relative to the closed surface may be determined. The aim of this paper is to present the underlying principle for the point location problem and not to provide an efficient algorithmic implementation.

2. A Physical Principle Underlying the Point Location Algorithm

According to the electromagnetic field theory, a point charge gives rise to an electric field whose flux is measured by the number of lines of force that cut through a surface. Furthermore, Gauss' Law [11] states that for any closed hypothetical surface in 3D-space, the flux through the surface is related to the net charge, q, enclosed by the surface through a surface integral, shown in Equation 1.

$$\epsilon_0 \oint \vec{E}.\vec{dS} = q, \tag{1}$$

where ϵ_0 is the permittivity constant, $\vec{E}$ is the vector electric field at a point on the surface, and $\vec{dS}$ denotes the outward normal vector to the surface at that point. Where the net charge enclosed by the volume corresponding to a hypothetical surface is zero, the flux through the surface is also zero. Otherwise, the net flux through the surface is non-zero.

The key elements in Gauss' Law are that it applies to any arbitrary closed surface in 3D-space and that the flux through the surface is related to the net charge enclosed by the surface. Clearly, the net charge enclosed by a surface may be considered a point charge, without any loss in generality. Assume a positive point charge located at a point P with respect to an arbitrary closed surface, S, in 3D-space. P may be either outside the surface or inside the surface. Where P lies on the surface, the issue of point location is easily settled by examining whether the coordinates of the point satisfy the mathematical equation of the surface.

First, consider that P is located outside S. Although the arguments in this paper apply to any arbitrary surface, for simplicity, assume that S is either a spherical surface or a U-shaped rectangular cylinder, as shown in Figure 5. According to Gauss' Law, the net flux through S due to P must be zero. To satisfy this requirement, Feynman [12] argues that, for any line of force, γ, emanating at P and terminating at infinity, if it intersects S, the net flux through S, due to γ, must be zero. According to Feynman [12], any volume can be thought of as completely made up of infinitesimal truncated cones with the apex at P. For each line of force, represented by E, the infinitesimal truncated cone may correspond to one continuous unit as in the case of the sphere in Figure 5 or a set of disconnected truncated cones as in the case of the U-shaped rectangular cylinder also shown in Figure 5. Gauss' Law dictates that the flux of E entering the leading surface of any infinitesimal truncated cone must equal the flux of E exiting the corresponding trailing surface of the truncated cone, such that the net flux is zero. Feynman [12] explains it clearly by utilizing the argument that the intersecting end surfaces are infinitesimally small so that they subtend an infinitesimal angle from the source, and that the E field is sufficiently uniform over the surface such that we can use just its value at the center. Thus, the flux entering the sphere through the infinitesimal surface "a" must be equal to that exiting through the infinitesimal surface "b." Also, the flux entering the rectangular cylinder through infinitesimal surface "a" must be equal to that exiting through the infinitesimal surface "b" and that entering through "c" must equal the flux exiting through "d." Similarly, the flux through " a' " must cancel out that through " b' ".

Therefore, where E intersects the surface S, there must be an integral number of pairs of intersecting points, implying a total of an even number of intersection points. If, on the contrary, we assume that the number of intersection points is odd, then there is one intersection point through which the flux of E either enters or exits S and there is the absence of the corresponding intersection point to force the net flux through S to equate to zero. This would violate Gauss' Law. Therefore, the number of intersections of E with S must be even, zero included.

Next, consider that P is located inside S as shown for the three closed surfaces in Figure 4. As before, although the arguments in this paper apply to any arbitrary surface, for simplicity, assume that S is a closed sphere, a toroid, or a rectangular dumb-bell. Gauss' Law requires the net flux out of S to be positive. Feynman [12] argues that every line of force, γ, emanating at P must intersect S, at least once. Utilizing similar arguments as before, namely that any volume can be thought of as completely made up of infinitesimal truncated cones with the apex at P, for each line of force, represented by E, the corresponding infinitesimal truncated cone may either consist of one continuous unit as in the case of the sphere in Figure 4 or a set of disconnected truncated cones as in the case of the toroid and the rectangular dumb-bell, also shown in Figure 4. Since the point charge at P lies inside the surface, Gauss' Law implies that the net flux exiting the infinitesimal surface must be finite. Thus, for the sphere S, the flux exiting the infinitesimal surface "a" of the sphere must be finite. For the toroid and the rectangular dumb-bell, while the flux entering the infinitesimal surface "b" equals that exiting the surface "c," the flux exiting the surface "a" is finite, implying that the net flux exiting out of the surface is non-zero. Therefore, the number of intersections of any E line of force with the closed surface must be odd, at least 1, so that the outward flux through at least one infinitesimal surface is non-zero, yielding a net positive outward flux.

Thus, for any arbitrary closed surface in 3D-space, a straight line, originating at any point P and extending to infinity, must intersect the surface an odd number of times, at least 1, if P is located within the surface. Where P is located outside the surface, the straight line from P may either never intersect the surface or intersect it an even number of times. This constitutes the definition of "point location" for any arbitrary closed surface in 3D-space. Since Gauss' Law may be re-written for 2 dimensions using a line integral instead of a surface integral, the "odd parity" rule for planar point location also derives its basis from the electromagnetic field theory.

The polygon in Figure 3(a) poses an interesting challenge to the key thesis in this manuscript. Although the point P appears to lie within the polygon, according to the odd parity rule, any ray emanating at P, except those passing through the vertices including S, intersects the polygon twice. Therefore, P should lie outside the polygon. For a better understanding, consider that there are three polygons – the triangle SGF, the pentagon ASDCB, and the octagon ASGFSDCB. Figure 3(b) enables a better appreciation of the octagon AHGFEDCB, where H and E are apart by an infinitesimal distance. In truth, the point P lies in-

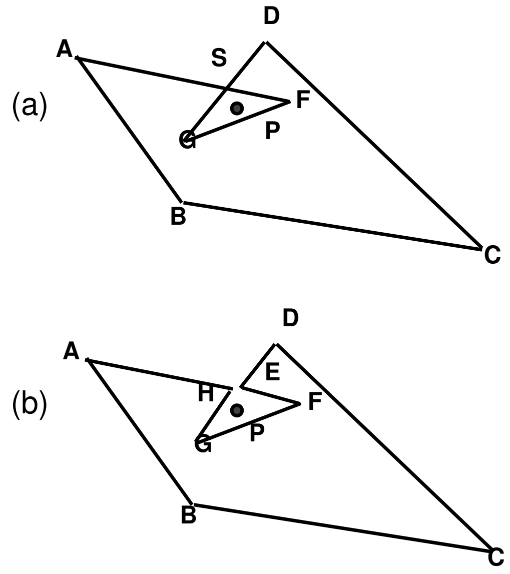

Figure 3: Point Location in a Complex Polygon in 2D-Space

side the triangle SGF. The point P also lies within the pentagon ASDCB. In both cases, any ray drawn from P will intersect the polygon only once. However, the point P lies outside the octagon ASGFSDCB in Figure 3(a) which is apparent more clearly in the octagon AHGFEDCB in Figure 3(b).

The above findings are corroborated by Gauss' Law, as follows. Assume, on the contrary, that the point P lies within the octagon and that it holds a positive charge. Therefore, the net flux through the octagon must be non-zero. Now, an electric field line emanating from P and terminating at infinity will intersect the octagon twice, except when it passes through S, exactly as in the case of the odd parity rule. The role of S is anomalous and, for the diagram in Figure 3(a) to be viewed as an octagon, S cannot constitute a vertex. Thus, an electric field line through S is not meaningful and the representation in Figure 3(b) is more appropriate. Assume that the flux flows relative to the octagon at these intersecting points are given by F1 and F2, respectively. Also, assume arbitrarily that a positive value implies flux exiting the octagon, while a negative value implies flux entering the octagon. Since P is assumed to lie within the octagon, F1 must be positive, i.e. the ray emanating from P must first exit the octagon. The quantity, F2, cannot assume a positive value since once the ray has exited the octagon, it cannot exit again without first entering it. Therefore, F2 must assume a negative sign. Thus, the signs of F1 and F2 are opposite, and utilizing Feynman's [12] argument for an infinitesimal cone of flux, F1 and F2 will cancel each other, implying that the net flux through the octagon is zero. This clearly contradicts Gauss' Law. Since Gauss' Law is a fundamental physical law that underlies the electromagnetic field theory, it cannot be violated. Thus, the point P must lie outside the octagon and no anomaly is implied between the odd parity rule and Gauss' Law.

Although it is not the aim of this paper to present data structures and algorithms for point location, the computation required to locate any given point relative to a hypothetical surface in 3D-space, is presented as follows. First the equation of the surface is developed. Then, the coordinates of the point are substituted to verify whether they satisfy the equation of the surface. If affirmative, the point lies on the surface. Otherwise, the point is located either within or outside the surface. Since the fundamental principle applies to any line of force, emanating at the positive point charge, a line is constructed to pass through the given point and the origin and its equation is synthesized. Next, the intersections, if any, between the line and the surface are obtained by solving for sets of $\{$x,y,z$\}$ values that simultaneously satisfy both of the equations. Where the set of $\{$x,y,z$\}$ values is nil, the point is considered to lie outside the surface. Otherwise, the magnitude and signs of the distances – P_1I_1, P_1I_2, ..., P_1I_j, from P to all of the intersection points, $\{$1,2,...,j$\}$, along the straight line through P, are computed. From this knowledge, it is deduced whether P is located inside or outside the closed surface depending on whether the number of intersections of the line originating at P and the surface are odd or even.

3. Conclusions

The literature presents a mechanism for planar point location relative to a polygon, as proposed by Sutherland and Hodgman. Although it is rich in optimal data structures and efficient algorithms for fast execution of the planar point location problem, a proof of correctness for the general problem of point location relative to any arbitrary surface in 3D-space is absent in the literature. This paper has argued that the electromagnetic field theory and Gauss's Law constitute a fundamental basis for the "odd parity" rule and has correctly extended the "odd parity" rule to locate points relative to any arbitrary closed surface in 3D-space.

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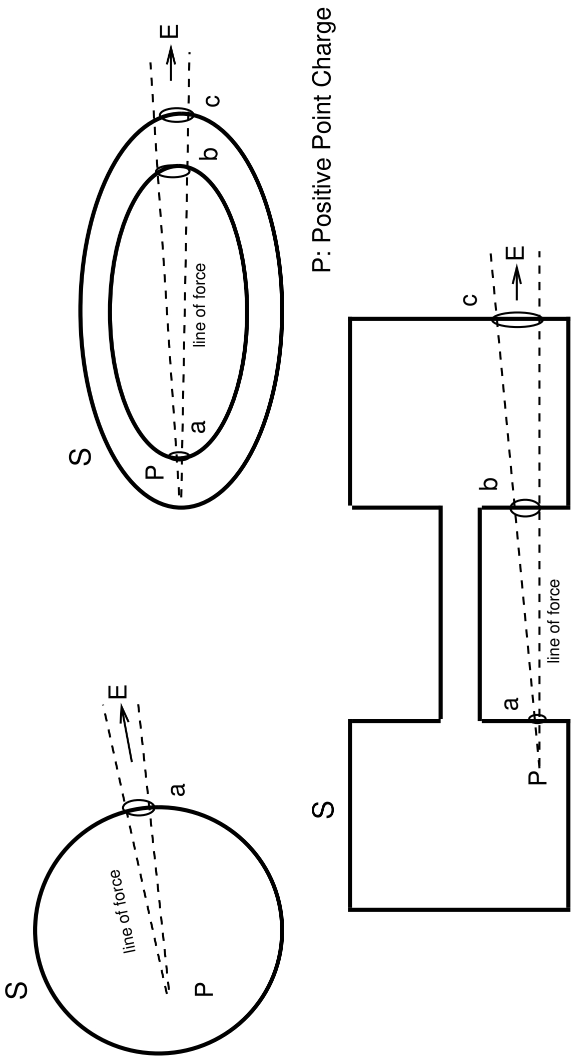

Figure 4: Electromagnetic Flux for a Positive Point Charge Located Inside a Surface in 3D-space

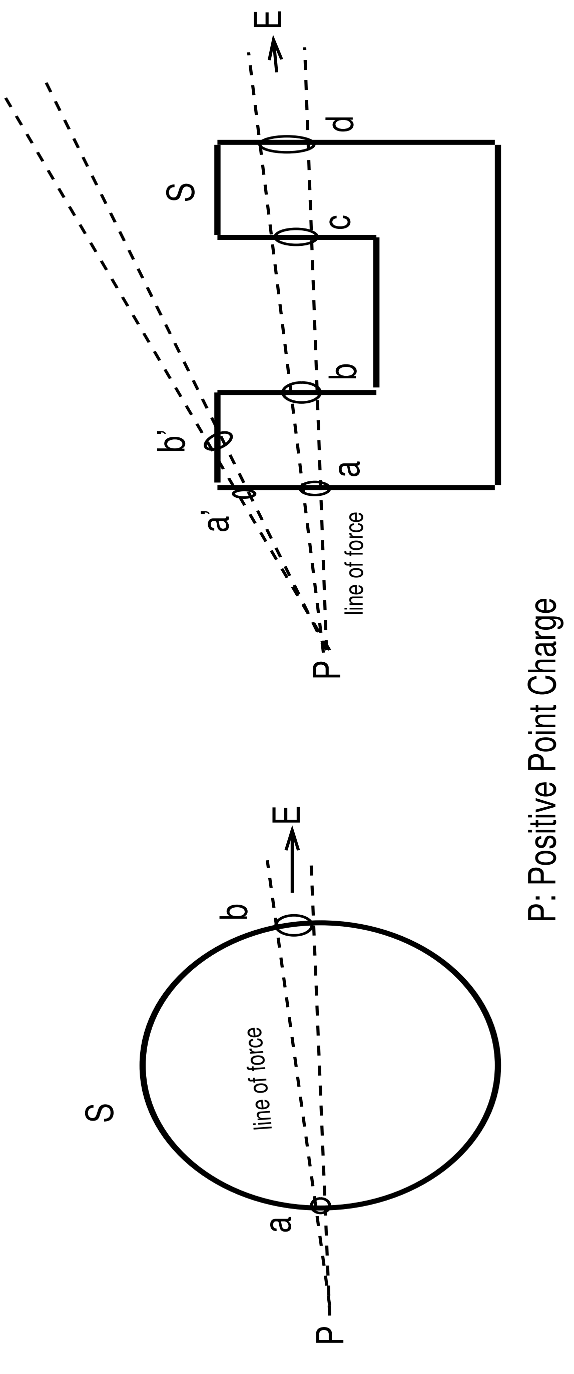

Figure 5: Electromagnetic Flux for a Positive Point Charge Located Outside a Surface in 3D-space