# MODIFIED CONSTRUCTION OF THE RATIONAL BEZIER CURVE BASED ON THE BARYCENTRIC COORDINATES

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Abstract: A demonstration of the more extensive paper on the example of the planar rational Bezier curve is presented. Definition of a standard rational Bezier curve in $E^2$ is based on the homogeneous affine coordinates i.e. on the projection of the 3D space into the plane $\rho \equiv z - 1 = 0$. The rational Bezier curves have been modified so that, the homogeneous affine coordinates have been replaced by the homogeneous barycentric coordinates and the plane $\rho \equiv z - 1 = 0$ by the plane $\rho \equiv \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$ where $p, q, r \in R - \{0\}$.

Keywords: Bezier curve, projection, affinity

## 1. Introduction

The planar Bezier curves are studied i.e. curves in the Euclidean plane $E^2$. Standard construction of these curves is based on 3D considerations. So, we assume that $E^3$ is given Euclidean space and $\langle O, \overline{e}_1, \overline{e}_2, \overline{e}_3 \rangle$ is a fixed Cartesian coordinate system of this space. Moreover, we assume that in $E^3$ there are given two copies $\pi$ and $\rho$ of the plane $E^2$, where plane $\rho$ is the result of the rotation (proper or improper) of the plane $\pi$ about the axis $\pi \cap \rho$. We suppose also that in each of these planes the Cartesian reference frame is choosen so that one is an image of the other under above rotation. Then each ordered couple $[x, y]$ of real numbers (point of $E^2$) can be considered as a point of the plane $\pi[\rho]$ with coordinates $[x, y]$ according to the above mentioned reference frame of $\pi[\rho]$.

## 2. Standard Rational Bezier Curve

In this case, the planes $\pi \equiv z = 0$ with the Cartesian coordinate system $\langle O, \overline{e}_1, \overline{e}_2 \rangle$ and $\rho \equiv z - 1 = 0$ with the Cartesian coordinate system $\langle P, \overline{a}_1, \overline{a}_2 \rangle$, where $\overline{a}_1 = \overline{e}_1, \overline{a}_2 = \overline{e}_2$ and $P = [0, 0, 1]$, are considered as the copies of the plane $E^2$.

An $n$-th degree planar rational Bezier curve $R$ is defined by

$$R(t)=\frac{\sum_{i=0}^{n} w_i V_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}=\left(\frac{\sum_{i=0}^{n} x_i w_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)},\frac{\sum_{i=0}^{n} y_i w_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}\right)^{\top} \tag{1}$$

where $V_i, i=0,1,\ldots,n$ are given points of $E^2$ with coordinates $[x_i, y_i], w_i, i=0,1,\ldots,n$ are positive real numbers and $B_i^n(t)$ are the Bernstein polynomials. The $w_i$ are called weights of the Bezier points $V_i$. The restriction $w_i > 0$ ensures that the denominator in (1) is strictly positive for $t \in \langle 0,1\rangle$. By setting all weights $w_i = 1$ in (1) we obtain standard nonrational Bezier curve. The construction of the standard Bezier curve (1) can be described by these steps.

To the points $V_i, i=0,1,\ldots,n$, we can assign their homogeneous coordinates $[x_i, y_i, 1]$. These triples can be interpreted as 3D coordinates of the orthogonal projections $V_i^{\perp}$ of the points $V_i \in \pi$ into the plane $\rho \equiv z-1=0$. On the other hand, to the points $V_i, i=0,1,\ldots,n$ and their weights $w_i$ can be assigned another triples $[w_i x_i, w_i y_i, w_i]$, their homogeneous coordinates. These triples can be interpreted as 3D coordinates of some points $\overline{V_i}$ on the lines $OV_i^{\perp}$, $i=0,1,\ldots,n$. The points $\overline{V_i}, i=0,1,\ldots,n$ as Bezier points determine the 3D nonrational Bezier curve

$$\overline{R}(t)=\sum_{i=0}^{n}\overline{V_i}B_i^n(t)=\begin{pmatrix}\sum_{i=0}^{n} w_i x_i B_i^n(t)\\ \sum_{i=0}^{n} w_i y_i B_i^n(t)\\ \sum_{i=0}^{n} w_i B_i^n(t)\end{pmatrix}=:\begin{pmatrix}S_x\\ S_y\\ S_1\end{pmatrix} \tag{2}$$

The image $R$ of the curve $\overline{R}(t)$, under the projection with centre $O=[0,0,0]$ and the image plane $\rho \equiv z-1=0$, has a 3D analytical expression

$$\left[\frac{\sum_{i=0}^{n} w_i x_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)},\frac{\sum_{i=0}^{n} w_i y_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)},1\right]=\left[\frac{S_x}{S_1},\frac{S_y}{S_1},1\right]=\left[\frac{\sum_{i=0}^{n} w_i V_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)},1\right]$$

Now it is obvious, that (1) is the analytical representation of the image $R$ with respect to the 2D Cartesian coordinate system $\langle P,\overline{a}_1,\overline{a}_2\rangle$ of the plane $\rho$ or its orthogonal projection into the plane $\pi$ with respect to the Cartesian coordinate system $\langle O,\overline{e}_1,\overline{e}_2\rangle$.

Evidently, using the plane $\rho \equiv z-r=0$ instead of the plane $z-1=0$ leads up to the same curve. Does this result hold for the planes $\rho \equiv \frac{x}{p}+\frac{y}{q}+\frac{z}{r}-1=0$, too?

## 3. Modified Rational Bezier Curve

In this case, the planes $\pi \equiv z=0$ with Cartesian coordinate system $\langle O,\overline{e}_1,\overline{e}_2\rangle$ and $\rho \equiv \frac{x}{p}+\frac{y}{q}+\frac{z}{r}-1=0$ (i.e. $\rho \equiv XYZ$ for the points $X=[p,0,0], Y=[0,q,0]$ and $Z=[0,0,r]$) with the Cartesian coordinate system

$\langle P, \overline{a}_1, \overline{a}_2 \rangle$ where

$$\overline{a}_1 = \left[\frac{p^2q^2 + p^2r^2 \mp pqS}{S^2 \mp pqS}, \frac{-pqr^2}{S^2 \mp pqS}, \frac{-pq^2r \pm qrS}{S^2 \mp pqS}\right],$$

$$\overline{a}_2 = \left[\frac{-pqr^2}{S^2 \mp pqS}, \frac{p^2q^2 + q^2r^2 \mp pqS}{S^2 \mp pqS}, \frac{-p^2qr \pm prS}{S^2 \mp pqS}\right]$$

and

$$P = \left[\frac{pq^2r^2}{S^2 \mp pqS}, \frac{p^2qr^2}{S^2 \mp pqS}, \frac{p^2qr \mp pqrS}{S^2 \mp pqS}\right]$$

with $S = \sqrt{p^2q^2 + p^2r^2 + q^2r^2}$, are considered as the copies of the plane $E^2$. It is obvious, that the reference frame $\langle P, \overline{a}_1, \overline{a}_2 \rangle$ is the result of the rotation of the frame $\langle O, \overline{e}_1, \overline{e}_2 \rangle$ about the axis $o \equiv XY$ into the plane $\rho$. So, there have been defined two Cartesian coordinate frames in the plane $\rho$, one for sign $-$ and other for sign $+$. These Cartesian coordinate frames are symmetric with respect to the line $o \equiv XY$.

In the special case, for the plane $\rho \equiv x + y + z - 1 = 0$ we obtain

$$(a)\ \ \overline{a}_1 = \frac{1}{\sqrt{3}}[-(p+1), p, 1], \overline{a}_2 = \frac{1}{\sqrt{2}}[p, -(p+1), 1], P = -\frac{1}{\sqrt{3}}[p, p, 1]$$

for sign $-$, or

$$(b)\ \overline{a}_1 = \frac{1}{\sqrt{3}}[-p, p+1, -1], \overline{a}_2 = \frac{1}{\sqrt{3}}[p+1, -p, -1], P = \frac{1}{\sqrt{3}}[-(p+1), -(p+1), 1]$$

for sign $+$, where $p = \frac{-(1+\sqrt{3})}{2}$.

Now, to the points $V_i = [x_i, y_i]$, or $V_i = [x_i, y_i, 0], i = 0, 1, \ldots, n$, we can assign their orthogonal projections into the plane $\rho$:

$$V_i^{\perp} = \left[x_i + \frac{pq^2r^2}{S^2}(1 - \frac{x_i}{p} - \frac{y_i}{q}), y_i + \frac{p^2qr^2}{S^2}(1 - \frac{x_i}{p} - \frac{y_i}{q}), \frac{p^2q^2r}{S^2}(1 - \frac{x_i}{p} - \frac{y_i}{q})\right]$$

Then the ordered triples

$$\left[\frac{x_i}{p} + \frac{q^2r^2}{S^2}(1 - \frac{x_i}{p} - \frac{y_i}{q}), \frac{y_i}{q} + \frac{p^2r^2}{S^2}(1 - \frac{x_i}{p} - \frac{y_i}{q}), \frac{p^2q^2}{S^2}(1 - \frac{x_i}{p} - \frac{y_i}{q})\right]$$

can be taken as *the barycentric coordinates of the points* $V_i^{\perp}$ *with respect to the triangle* $XYZ$. By the weights $w_i$, we can assign to these triples the following triples of the homogeneous barycentric coordinates of the points $V_i^{\perp}$:

$$\left[\frac{x_i}{p}w_i + \frac{q^2r^2}{S^2}\alpha_i w_i, \frac{y_i}{q}w_i + \frac{p^2r^2}{S^2}\alpha_i w_i, \frac{p^2q^2}{S^2}\alpha_i w_i\right],$$

where $\alpha_i = 1 - \frac{x_i}{p} - \frac{y_i}{q}$.

On the other hand, these triples can be considered as 3D coordinates of some points $\overline{V_i}$. So, by the points $V_i \in E^2$ and the weights $w_i \in R$, we have defined in $E^3$ the points

$$\overline{V_i} = \left[\frac{x_i}{p}w_i + \frac{q^2r^2}{S^2}\alpha_i w_i, \frac{y_i}{q}w_i + \frac{p^2r^2}{S^2}\alpha_i w_i, \frac{p^2q^2}{S^2}\alpha_i w_i\right], i = 0, 1, \ldots, n$$

These points as Bezier points define 3D Bezier curve

$$\overline{P}(t) = \sum_{i=0}^{n} \overline{V_i} B_i^n(t) = \sum_{i=0}^{n} \begin{pmatrix} \frac{x_i}{p}w_i + \frac{q^2r^2}{S^2}\alpha_i w_i \\ \frac{y_i}{q}w_i + \frac{p^2r^2}{S^2}\alpha_i w_i \\ \frac{p^2q^2}{S^2}\alpha_i w_i \end{pmatrix} B_i^n(t)$$

i.e.

$$\overline{P}(t) = \begin{pmatrix} \frac{1}{p}\sum_{i=0}^{n} x_i w_i B_i^n(t) + \frac{q^2r^2}{S^2}\sum_{i=0}^{n} \alpha_i w_i B_i^n(t) \\ \frac{1}{q}\sum_{i=0}^{n} y_i w_i B_i^n(t) + \frac{p^2r^2}{S^2}\sum_{i=0}^{n} \alpha_i w_i B_i^n(t) \\ \frac{p^2q^2}{S^2}\sum_{i=0}^{n} \alpha_i w_i B_i^n(t) \end{pmatrix}$$

or

$$\overline{P}(t) = \begin{pmatrix} \frac{1}{p}S_x + \frac{q^2r^2}{S^2}[S_1 - \frac{1}{p}S_x - \frac{1}{q}S_y] \\ \frac{1}{q}S_y + \frac{p^2r^2}{S^2}[S_1 - \frac{1}{p}S_x - \frac{1}{q}S_y] \\ \frac{p^2q^2}{S^2}[S_1 - \frac{1}{p}S_x - \frac{1}{q}S_y] \end{pmatrix} \tag{3}$$

The entries of the right-hand side matrix of this equation can be considered as homogeneous barycentric coordinates of the point $\overline{P}(t)$ with sum $\sum_{i=0}^{n} w_i B_i^n(t)$. Its nonhomogeneous barycentric coordinates are obtained as 3D coordinates of the projection with the centre $O$ of the point $\overline{P}(t)$ into the plane $x + y + z - 1 = 0$. These coordinates are given by

$$\frac{1}{S_1}\left[\frac{1}{p}S_x + \frac{q^2r^2}{S^2}S_{1xy}, \frac{1}{q}S_y + \frac{p^2r^2}{S^2}S_{1xy}, \frac{p^2q^2}{S^2}S_{1xy}\right],$$

where $S_{1xy} = S_1 - \frac{1}{p}S_x - \frac{1}{q}S_y$. Then

$$\frac{1}{S_1}\left[S_x + \frac{pq^2r^2}{S^2}S_{1xy}, S_y + \frac{p^2qr^2}{S^2}S_{1xy}, \frac{p^2q^2r}{S^2}S_{1xy}\right] \tag{4}$$

are 3D coordinates of the projection $P(t)$ of the point $\overline{P}(t)$ into the plane $\rho \equiv \frac{x}{p} + \frac{y}{q} + \frac{z}{r} - 1 = 0$.

There exists also another way that leads up to the same result $P(t)$ i.e. $P(t)$ is projection with the centre $O$ of the 3D Bezier curve generated by the Bezier points "$w_i V_i^{\perp}$" into the plane $\rho$.

Using the known equations for the transformation of the 3D Cartesian coordinate systems are obtained 2D coordinates of the point $P(t)$ in the Cartesian coordinate system $\langle P, \overline{a}_1, \overline{a}_2 \rangle$ in the form

$$\begin{aligned} x_P &= \tfrac{S_x}{S_1} - \tfrac{pq^2r^2}{S^2 \mp pqS}\left[\tfrac{1}{p}\tfrac{S_x}{S_1} + \tfrac{1}{q}\tfrac{S_y}{S_1} - 1\right], \\ y_P &= \tfrac{S_y}{S_1} - \tfrac{p^2qr^2}{S^2 \mp pqS}\left[\tfrac{1}{p}\tfrac{S_x}{S_1} + \tfrac{1}{q}\tfrac{S_y}{S_1} - 1\right] \end{aligned} \tag{5}$$

From the equations (5) can be derived the following analytical representation of the curve $P(t)$ in the Cartesian coordinate system $\langle P, \overline{a}_1, \overline{a}_2 \rangle$ of the plane $\rho$

$$P(t) = \frac{\sum_{i=0}^{n} P_i w_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}, \tag{6}$$

where

$$P_i = \begin{pmatrix} x_i - \frac{pq^2r^2}{S^2 \mp pqS}\left(\frac{x_i}{p} + \frac{y_i}{q} - 1\right) \\ y_i - \frac{p^2qr^2}{S^2 \mp pqS}\left(\frac{x_i}{p} + \frac{y_i}{q} - 1\right) \end{pmatrix}.$$

It is obvidious, that $P(t)$ represents the rational Bezier curve.

*This curve will be our modified rational Bezier curve.*

From the formulas (6), it is obvious, that $P(t)$ is the image of the rational Bezier curve $R'(t)$ of the plane $\rho$ which Bezier control points have the same coordinates $[x_i, y_i]$ in $\langle P, \overline{a}_1, \overline{a}_2 \rangle$ as the points $V_i$ in $\langle O, \overline{e}_1, \overline{e}_2 \rangle$ under an affine transformation

$$\begin{aligned} \varphi: \quad x' &= x - \tfrac{pq^2r^2}{S^2 \mp pqS}\left(\tfrac{x}{p} + \tfrac{y}{q} - 1\right) \\ y' &= y - \tfrac{p^2qr^2}{S^2 \mp pqS}\left(\tfrac{x}{p} + \tfrac{y}{q} - 1\right) \end{aligned} \tag{7}$$

of the plane $\rho$. Because of construction of $\langle P, \overline{a}_1, \overline{a}_2 \rangle$, the curve $R'(t)$ is evidently a result of the rotation of the standard rational Bezier curve $R(t) \in \pi$ about the axis $o \equiv XY$ into the plane $\rho$.

From the equations (5), this fact can be also derived, because $\left[\frac{S_X}{S_1}, \frac{S_y}{S_1}\right]$ are the coordinates of the curve $R(t)$ in the $\langle O, \overline{e}_1, \overline{e}_2 \rangle$ and the rotated curve $R'(t)$ in $\langle P, \overline{a}_1, \overline{a}_2 \rangle$, too.

The affine transformation $\varphi$ is an orthogonal affinity with axis $o \equiv \frac{x}{p} + \frac{y}{q} - 1 = 0$ (i.e. $o \equiv XY$) and the coefficient $k = \mp \frac{pq}{S}$.

In the special case, for the plane $\rho \equiv x + y + z - 1 = 0$ the equations (7) have the following forms

$$x' = \frac{1}{3}[(p+2)x + (p-1)y - (p-1)]$$

$$y' = \frac{1}{3}[(p-1)x + (p+2)y - (p-1)]$$

or

$$x' = \frac{1}{3}[-(p-1)x - (p+2)y + (p+2)]$$
$$y' = \frac{1}{3}[-(p+2)x - (p-1)y + (p+2)]$$

and represent the orthogonal affinity with axis $o \equiv x + y - 1 = 0$ and coefficient $-\frac{1}{\sqrt{3}}$ or $\frac{1}{\sqrt{3}}$.

For $p \to \infty$ and $q \to \infty$ (and convenient sign) in the formulas (7) i.e. for $\rho \equiv z - r = 0$ or $\rho \equiv z - 1 = 0$, we obtain identical transformation $x' = x$ and $y' = y$ i.e. $P(t) = R'(t) \equiv R(t)$.

So, if in standard construction of the rational Bezier curve the plane $\rho \equiv z - 1 = 0$ is replaced by the plane $\rho \equiv \frac{x}{p} + \frac{y}{q} + \frac{z}{r} - 1 = 0$, a rational Bezier curve in the plane $\rho$ will be obtained too. This curve is the image of the curve $R(t)[R'(t)]$ in the orthogonal affinity of the plane $\rho$ with axis $o \equiv \frac{x}{p} + \frac{y}{q} - 1 = 0$ and coefficient $k = \mp\frac{pq}{S}$, where $S = \sqrt{p^2q^2 + p^2r^2 + q^2r^2}$.

Precisely, each modified rational Bezier curve can be considered as the image of the standard rational Bezier curve in the orthogonal affinity which coefficient $k$ satisfies the conditions $|k| < 1, k \neq 0$. Conversely, each image of the standard rational Bezier curve in the orthogonal affinity for example with the axis $o \equiv \frac{x}{p} + \frac{y}{q} - 1 = 0$ $(O \notin o)$ and coefficient $k$ satisfying the conditions $|k| < 1, k \neq 0$, can be considered to be a modified rational Bezier curve corresponding to the plane $\rho \equiv \frac{x}{p} + \frac{y}{q} + \frac{z}{r} - 1 = 0$, where $r$ is a root of the equation

$$r^2 = \frac{1-k^2}{k^2} \cdot \frac{p^2q^2}{p^2+q^2} \tag{8}$$

Evidently, if the plane $\rho$ is replaced by the plane $\rho' \equiv \frac{x}{\gamma p} + \frac{y}{\gamma q} + \frac{z}{\gamma r} - 1 = 0$, $\gamma \neq 0$, then the new axis $o'$ will be parallel to the axis $o$, but the coefficient $k$ will be unchanged. Especially, if we replace $p \to -p$, $q \to -q$, $r \to -r$, the axis $o'$ will be the image of the axis $o$ under the symmetry with the centre $O$. The single change of the parameter $r$ has no influence on the axis $o$, but it changes in general the quantity of $S$ and so the coefficient $k$ (in the special case $r \to -r$, the coefficient $k$ will be unchanged too).

Finally, it is obvious, that for positive numbers $p, q$ and $k$, $0 < k < 1$, the equations (7) of the affinity $\varphi$ can be rewritten as follows

$$\begin{aligned} x' &= x - \tfrac{1}{p}(1-k)\tfrac{p^2q^2}{p^2+q^2}(\tfrac{x}{p} + \tfrac{y}{q} - 1) \\ y' &= y - \tfrac{1}{q}(1-k)\tfrac{p^2q^2}{p^2+q^2}(\tfrac{x}{p} + \tfrac{y}{q} - 1) \end{aligned} \tag{9}$$

The geometrical interpretation of our construction is based on these equations [or on equations (7)]. It immediately follows from geometrical interpretation of the basic elements of the affinity $\varphi$. More precissely, the coefficient $k$ can be interpreted as a scalling factor in the direction orthogonal to the axis $o$.

Because of $|k| = \frac{1}{\sqrt{1+R^2r^2}}$, where $R^2 = \frac{p^2+q^2}{p^2q^2}$ is for fixed axis $o$ constant, the factor $k$ can be modelled by the parameter $r$ i.e. by the rotation of the plane $\rho$ around the axis $o$ so that its point $R = [0, 0, r]$ traces the $z$-axis. As $|r|$ increases through positive values (i.e. $|k| \to 0$), the curve $R(t)$ get closer and closer (by flattening) to the straight line segment on the axis $o$ and conversely.

## 4. Conclusions

The construction of the rational Bezier curve $P(t)$ has been described where the plane $\rho \equiv z - 1 = 0$ has been replaced by a more general plane $\rho \equiv \frac{x}{p} + \frac{y}{q} + \frac{z}{r} - 1 = 0$. The parameters $p, q, r$ can be used as the global shape parameters. Because the 3D coordinates of the orthogonal projection of the arbitrary point $[x, y] \in \pi$ are given analogicaly as the 3D coordinates of the points $V_i^{\perp}$, using (4) it is obvious, that $P(t)$ is the orthogonal projection of the curve $R(t)$ into the plane $\rho$ (i.e. $R(t)$ is preimage of $P(t)$ under this projection). So, the completely construction of the rational Bezier curve based on the arbitrary plane $\rho, O \notin \rho$ and the homogeneous barycentric coordinates gives the same result as the standard construction based on the plane $\rho$ and the homogeneous affine coordinates. Further, a connection between the rotated image $R'(t)$ and orthogonal projection $P(t)$ of the curve $R(t) \in \pi$ is represented by affinity $\varphi$ (in the plane $\rho \equiv z - 1 = 0$ these curves are identical).

Evidently, if instead of the orthogonal projection into the plane $\rho$ an another (admissible) parallel projection $f$ into the plane $\rho$ is choosen, then $f[R(t)]$ will be considered as the image of the curve $R(t)$ $(R'(t))$ under a skew affinity of the plane $\rho$ with the axis $o \equiv XY$.

It is interesting, that if we start the construction of the curve $P(t)$ in the plane $\rho$ i.e. $[x_i, y_i]$ are understood as the coordinates of the Bezier points $V_i$ in the Cartesian coordinate system $\langle P, \overline{a}_1, \overline{a}_2 \rangle$ we will get $P(t) = R(t)$. So, symmetric construction of the rational Bezier curve based on the homogeneous barycentric coordinates gives the same result as the construction based on the homogeneous affine coordinetes.

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