Visualizing Dynamical Systems near Critical Points*

Helwig Löffelmann, Helmut Doleisch, and Eduard Gröller
E-Mail: {helwig, helmut, groeller}@cg.tuwien.ac.at

Institute of Computer Graphics, Vienna University of Technology, Austria,
WWW: http://www.cg.tuwien.ac.at/home/

Abstract

In this paper we present two visualization techniques. One uses the topological structure of the dynamical system near critical points to build an abstract description of the flow. The other places bunches of streamlets around the critical points to visualize the flow characteristics locally. Combining both methods a powerful visualization technique is present, since both the topological information as well as intuitive visual cues are provided.

Keywords: visualization, dynamical systems, critical points, vector field topology

1 Introduction

During the past decade visualization has been established as an important scientific discipline. Among its topics the visualization of flow data is one of the most settled categories. The visualization of dynamical systems, on the other hand, is strongly related to flow visualization, since flows as well as dynamical systems are characterized by vectorial information in n-dimensional space.

In this paper we first present two new techniques for the visualization of a continuous dynamical system in 3D space, which belong to two different approaches concerning the analysis of the flow data. Furthermore, we demonstrate that combining both methods yields better results.

1.1 Dynamical systems

Any set of scalar variables, which changes over time according to a specific set of rules, is called a *dynamical system*. A *continuous* dynamical system

*Color plates: http://www.cg.tuwien.ac.at/research/vis/dynsys/AdvFPViz/.

is given by a set of ordinary differential equations [3], whereas a *discrete* dynamical system is specified by difference equations. A dynamical system is called *time-dependent*, if its definition depends on time itself. If, on the other hand, the rules determining the dynamics are static over time, a *steady*, i.e., a *time-independent* system is given. In this paper we discuss the visualization of continuous, steady, three-dimensional dynamical systems, i.e., 3D flow fields, which analytically are described through a function $\mathbf{f}$ and parameters $\mathbf{p}$ as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \quad \mathbf{x} \in \mathbf{R}^3, \mathbf{p} \in \mathbf{R}^m \tag{1}$$

Dynamical systems usually are depicted in *phase space*, which is defined by associating each system variable to an axis of a coordinate system. In our case a three-dimensional Cartesian coordinate system is given as phase space. Solutions of a dynamical system are called *trajectories*. A trajectory $\mathcal{T}(t)$ is given by a specific seed value $\mathcal{T}_0$ and the following integral equation:

$$\mathcal{T}(t) = \mathcal{T}_0 + \int_{s=0}^{t} \mathbf{f}(\mathcal{T}(s), \mathbf{p})\, ds \tag{2}$$

1.2 Approaches to analyze a dynamical system

There are many ways to investigate a dynamical system. One way to classify these approaches is to look at the level of abstraction which is used for investigation:

A very abstract approach to analysis is the investigation of the topological skeleton of a dynamical system [4, 5]. The topology of behavior is built from characteristic elements, such as, for example, critical points and separatrices (see also Section 1.3). Once the topological structure of a dynamical system is derived, the long term evolution is known for all the points in phase space qualitatively. However, almost no quantitative information, such as, e.g., spatial extent of trajectories, is provided through this approach.

Spatially more concrete insights are gained from building the geometry of behavior. Location and shape of the topological entities are composed to a geometry within phase space [1, 10]. The topological structure is extended by the spatial attributes of its elements. Thereby at least some quantitative information is provided. Nevertheless, the geometry of behavior is still a rather dense description of the flow without flow details apart from characteristic structures.

Detailed (local) information is gained by directly visualizing the dynamics using stream lines, stream surfaces, particle systems, or other integral objects [15]. Selected trajectories are visualized through these techniques. This kind of visualization technique always lacks completeness.

As the previously described approaches have their advantages and disadvantages each (see Tab. 1), a combination of them seems quite logical. Most

	density	completeness	quantities
the topological structure	high	usually yes	almost non
the geometry of behavior	medium	potentially yes	just a few
selective direct visualization	regular	no	yes, details

Table 1: density and completeness of information, and amount of quantities, provided through different approaches of dynamical system analysis.

approaches found in literature, on the other hand, concentrate on either the one or the other.

In this paper we present two visualization techniques – one featuring the topological information and the other using direct visualization – and show how a combination of both yields better results.

1.3 Vector field topology and local analysis

A common procedure to extract the topology of a dynamical system [5] is to proceed in the following way [4]:

1. Identify all the critical points $\bar{\mathbf{x}}_i$ (also called fixed points, roots, etc.) by solving equation $\mathbf{f}(\bar{\mathbf{x}}_i, \mathbf{p}) = 0$.

2. Investigate the Jacobian matrix of $\mathbf{f}$ at the critical points, i.e., $\nabla\mathbf{f}|_{\bar{\mathbf{x}}_i}$. In the hyperbolic case this matrix of derivatives represents the major components of the flow near the critical point. Eigen values and eigen vectors intuitively describe the characteristics of the dynamics of $\mathbf{f}(\bar{\mathbf{x}}_i + \Delta, \mathbf{p})$, i.e., of the flow near $\bar{\mathbf{x}}_i$.

   Depending on the local flow characteristics, critical points are classified as attractors, repellors, or saddles. Focal fixed points exhibit a pair of conjugated complex eigen values, whereas nodes exhibit real eigen values only.

3. Search for higher order characteristic elements, such as, e.g., cycles $\bar{\mathcal{T}}_i(t)$ ($\bar{\mathcal{T}}_i(t+T) = \bar{\mathcal{T}}_i(t)$). Again perform an analysis of derivatives to learn about local flow characteristics.

4. Extend the eigen manifolds near the characteristic elements into phase space to determine their relation to other characteristic elements. Thereby structures such as separatrices, i.e., elements which divide the phase space into regions of qualitatively different dynamics, are identified.

Extracting location and shape of the topological elements, the geometry of behavior [1] is constructed. In the next section we briefly review visualization techniques which communicate the geometry of behavior as an abstract description of the flow dynamics to the viewer.

1.4 Flow topology and visualization, state of the art

Helman and Hesselink [10] proposed to visualize the geometry of the topological structure of flow dynamics. Stream lines along the eigen vectors of critical points were used to show separatrices. Icons composed of line segments and small disks encoded the Jacobian matrix near fixed points. Globus et al. [9] came up with a tool to identify topological elements within data that is given at a discrete grid. Happe and Rumpf [17] extended the use of icons for representing local flow characteristics near fixed points of the system. Overviews of work in this area are given by Levit in 1992 [11] and Asimov et al. in 1995 [6].

Several other approaches to flow visualization are related to the visualization of vector field topology. The visualization of local properties [7, 13, 14, 16], for example, is an extension to the approach of visually encoding the Jacobian matrix. The visualization of tensor field topology [8] is also strongly related to this topic.

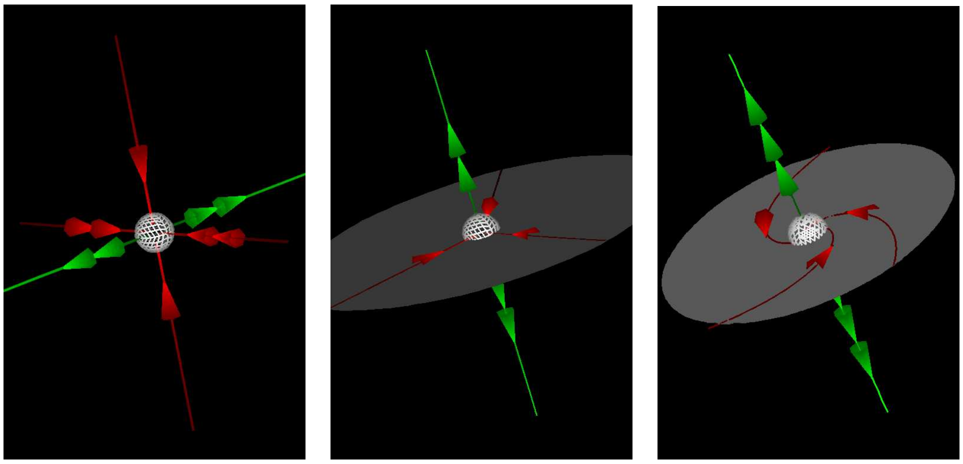
2 Visualizing the vicinity of critical points

In the following sections we describe two techniques to visualize the dynamics in the vicinity of critical points. One belongs to the geometry of behavior approach (see Section 1.2), whereas the other technique features direct visualization. In combination both techniques bring together the advantages of both approaches and give an expressive visualization technique.

2.1 CharDirs – visualizing the eigen manifolds

The first visualization technique presented here is similar to the iconic representation of flow near critical points [10, 17]. Instead of using a glyph to encode the flow topology, we directly represent the geometry of behavior by the use of stream lines and stream surfaces. We first inspect the eigen values of the Jacobian matrix and distinguish between topological different cases:

1. If all the eigen values are real, different from each other, and different from zero, three pairs of stream lines are integrated into the direction of the corresponding eigen vectors. Thereby the (locally) most significant trajectories are depicted (see Fig. 1(a)).



(a) (b) (c)

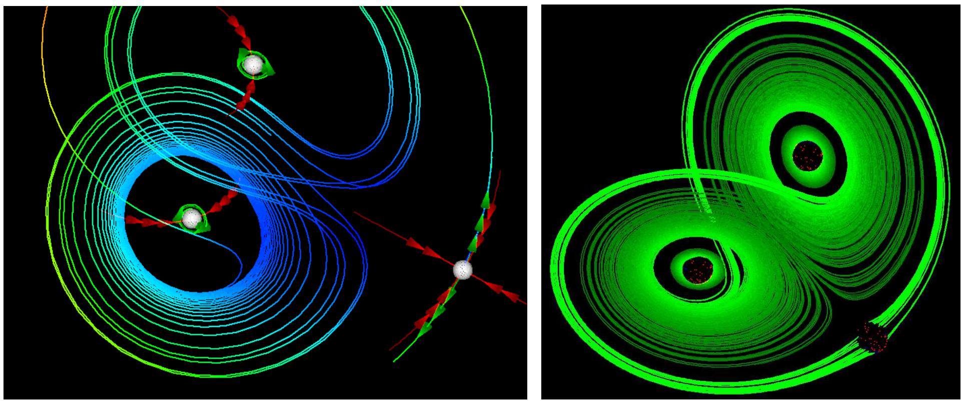
Figure 1: Visualizing the geometry of behavior near the critical point of a linear dynamical system – three different saddle configurations.

2. If all the eigen values are real, different from zero, but two of them are equal, a 1-manifold and a 2-manifold corresponding to the double eigen value build up the geometry of behavior near the critical point. In addition to a pair of stream lines we use three stream lines within the 2-manifold plus an optional stream surface to encode this special flow topology (see Fig. 1(b)).

3. If two eigen values are complex and the real parts of all eigen values are different from zero, the same geometry of behavior is present as in the second case. Thus the same visualization technique is used (see Fig. 1(c)). However, the flow characteristics are different – spiraling vs. radial attraction/repulsion.

To add more quantitative information we encode the order of magnitude of the eigen values by a certain number of arrows along the characteristic trajectories. Thereby the geometry of behavior near critical points is visualized for the most important cases. Degeneracies of flow geometry, e.g., non-hyperbolic fixed points, are not considered through this approach.

In Fig. 2(a), for example, the Lorenz system was visualized by the use of this technique. Two saddle foci with (each) a pair of conjugated complex eigen values and a large negative real eigen value drive the rotating characteristic of this dynamical system. A third saddle coordinates the alternating dominance of these two foci.



(a) (b)

Figure 2: Visualizing the flow characteristics near the critical points of the Lorenz system – CharDirs (a) and SphereTufts (b).

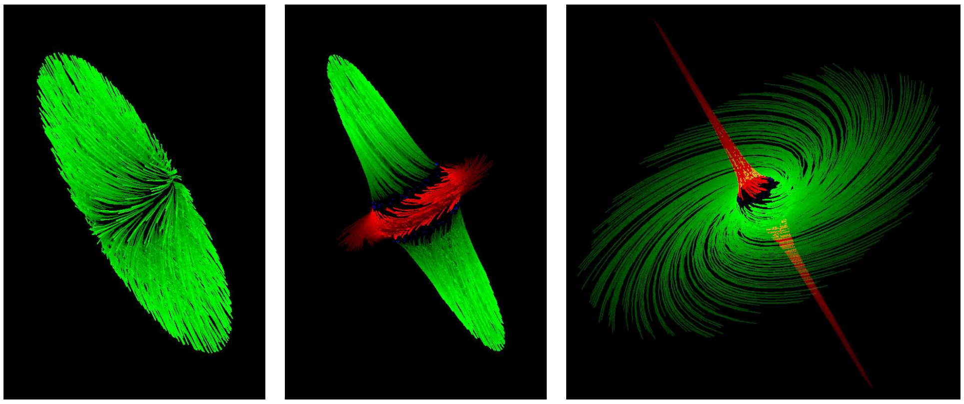
2.2 SphereTufts – using a bunch of streamlets

The second technique presented here is based on direct visualization of flow rather than on topological or geometrical analysis. A bunch of streamlets is used near the critical points of a dynamical system to directly represent the flow characteristics. The seed points of all the streamlets are stochastically chosen on a small sphere around the fixed point. Forward integration as well as backward integration in time is performed with a fixed length in time. Thereby the spatial length of the streamlets directly encodes velocity. Forward streamlets are colored different from backward integrated ones to distinguish between saddles from attractors/repellors.

In Fig. 3 three examples are given. Fig. 3(a) shows a linear repellor focus, whereas Fig. 3(b) depicts the saddle focus of a linear dynamical system. Fig. 3(c) shows another linear saddle focus with different eigen values. Using this direct visualization technique subtle differences in the flow characteristics become visible. For example, by the use of SphereTufts the differences between Fig. 3(b) and Fig. 3(c) become visible, although the flow geometry is identical in both cases: the flow component related to the real eigen value is much stronger in Fig. 3(c) than in Fig. 3(b).

In Fig. 2(b) the Lorenz system was visualized by placing bunches of streamlets near the critical points. The most important flow characteristics are intuitively depicted.

Although the results gained with this method are quite expressive, the uniform distribution of streamlet seed points might not be the optimal choice. Using a seed value distribution, which reflects the distance to the most characteristic directions, i.e., the trajectories which coincide with the eigen vectors of the critical point, the visualization is further improved. Un-



(a) (b) (c)

Figure 3: Visualizing a linear system near the critical point with a pair of conjugated complex eigen values: a repellor (a) and two saddles (b,c).

fortunately the implementation of this extension is not stable yet and thus no images are presented in this paper.

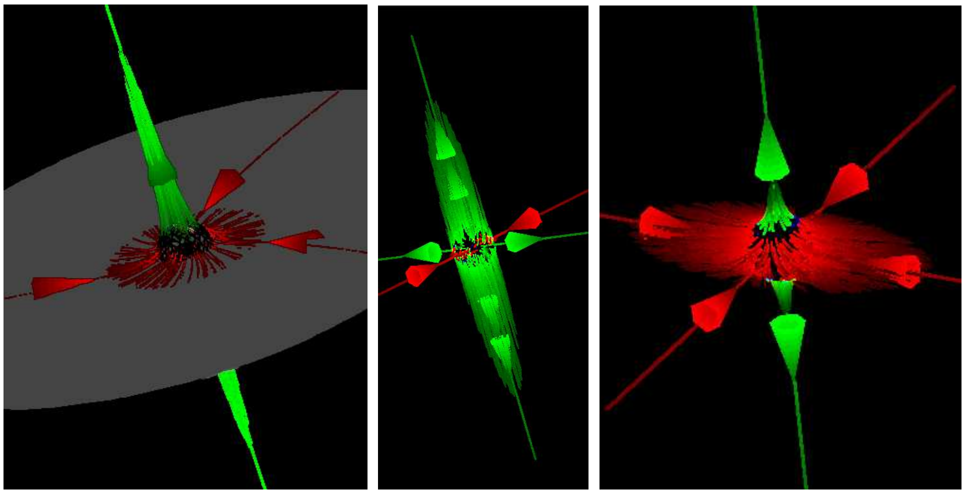
2.3 Combining CharDirs and SphereTufts

If both methods presented in the previous two sections are combined, the following advantages are achieved: first, a complete description of the flow topology is given through the CharDirs visualization. Additionally direct visualization cues are used to intuitively describe the local flow dynamics.

Fig. 4 demonstrates this combination of advantages in three different linear cases. A saddle focus (Fig. 4(a)) and two saddle nodes (Fig. 4(b,c)) are shown. Topology, geometry, and flow details are integrated within one pictorial representation. Fig. 5 shows the Lorenz attractor visualized with this combined technique. The strong differences concerning velocity (about three orders of magnitude) become visible through direct visualization, while still topological information is provided.

3 Implementation

Both techniques described above have been implemented into DynSys3D [12], a visualization system for dynamical systems based on AVS [2]. Due to the modular concept of DynSys3D, different visualization techniques can be directly combined with each other. Thereby the user easily composes the visualization configuration which is appropriate for a certain problem. Since visualization technique, numerical integrator, and dynamical system are quite independent modules within DynSys3D – they communicate



(a) (b) (c)

Figure 4: Three different saddle configurations visualized using both the eigen manifold visualization and bunches of streamlets.

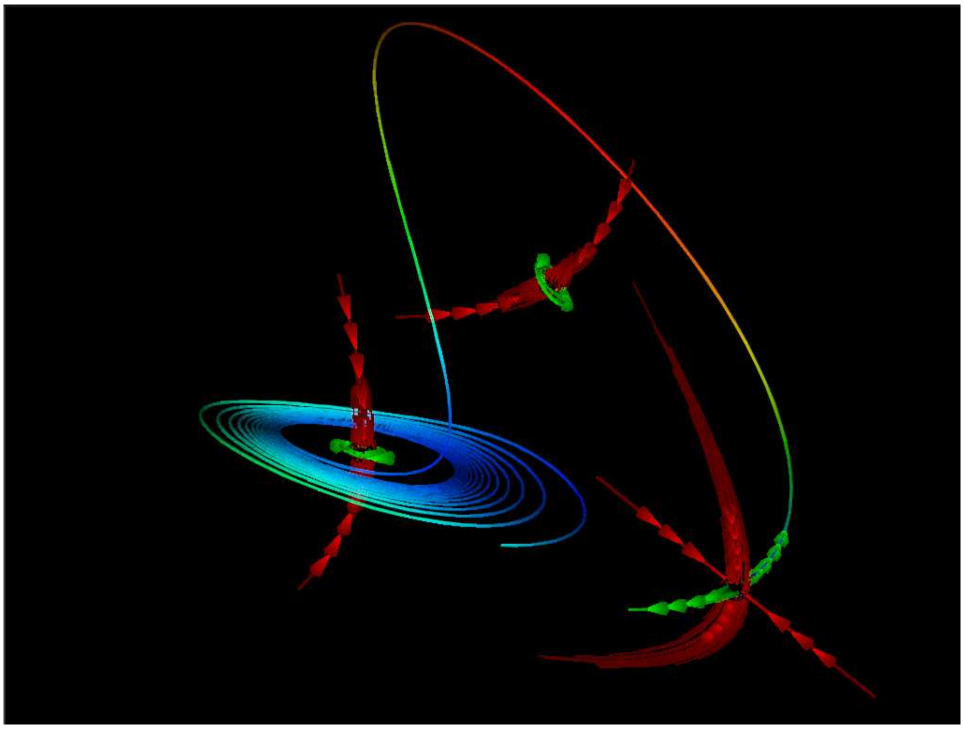


Figure 5: Visualizing the flow near the critical points of the Lorenz system – both bunches of streamlets and eigen manifold visualization were used.

through a narrow standard interface – new dynamical systems are easily integrated. The identification of fixed points and the evaluation of the Jacobian matrix have to be implemented.

4 Conclusions

Most approaches of either visualizing the geometry of behavior or, on the other hand, directly representing flow dynamics, have advantages and disadvantages (see Tab. 1). Visualizing the topology of behavior facilitates a dense and abstract way of entirely describing the flow dynamics. Directly visualizing the vector field yields more intuitive representations of the dynamical system by simultaneously lacking completeness.

We present two techniques belonging to both classes and demonstrate how a combination of both can be used to achieve a more expressive visualization method. Enhancing the abstract information by cues of direct visualization helps the viewer to more intuitively depict flow motion within critical regions.

Acknowledgements

The authors thank Lukas Mroz for supplying the stream surface module and Robert F. Tobler for his help with the final preparation of this paper.

References

[1] R. H. Abraham and C. D. Shaw. *Dynamics – The Geometry of Behavior*. Addison-Wesley, 2nd edition, 1992.

[2] Advanced Visual System Inc. *AVS Developers Guide, Release 4*, May 1992.

[3] V. I. Arnold. *Ordinary differential equations*. Springer, Berlin, 1992.

[4] D. K. Arrowsmith and C. A. Place. *An Introduction to Dynamical Systems*. Cambridge University Press, 1990.

[5] D. Asimov. Notes on the topology of vector fields and flows. Technical Report RNR-93-003, NASA Ames Research Center, 1993.

[6] D. Asimov, L. Hesselink, and F. Post. Visualization and topology of vector and tensor fields. Tutorial Notes #6, IEEE Visualization '95, 1995.

[7] W. C. de Leeuw and J. J. van Wijk. A probe for local flow field visualization. In *Proceedings of IEEE Visualization '93*, pages 39–45, 1993.

[8] T. Delmarcelle and L. Hesselink. The topology of second-order tensor fields. In *Proceedings of IEEE Visualization '94*, pages 140–148, 1994.

[9] A. Globus, C. Levit, and T. Lasinski. A tool for visualizing the topology of three-dimensional vector fields. In *Proceedings IEEE Visualization '91*, pages 33–40, 1991.

[10] J. Helman and L. Hesselink. Visualizing vector field topology in fluid flows. *IEEE Computer Graphics & Applications*, 11(3):36–46, 1991.

[11] C. Levit. Visualizing the topology of vector fields – an annotated bibliography. Technical Report RNR-92-032, NASA Ames Research Center, 1992.

[12] H. Löffelmann and E. Gröller. DynSys3D: A workbench for developing advanced visualization techniques in the field of three-dimensional dynamical systems. In *Proceedings of WSCG '97*, pages 301–310, Plzen, Czech Republic, February 1997.

[13] H. Löffelmann and E. Gröller. Enhancing the visualization of characteristic structures in dynamical systems. Technical Report TR-186-2-98-05, Institute of Computer Graphics, Vienna University of Technology, Austria, January 1998.

[14] H. Löffelmann, Z. Szalavári, and E. Gröller. Local analysis of dynamical systems – concepts and interpretation. In *Proceedings of WSCG '96*, pages 170–180, Plzen, Czech Republic, February 1996.

[15] F. H. Post and T. van Walsum. Fluid flow visualization. In H. Hagen, H. Müller, and G. M. Nielson, editors, *Focus on Scientific Visualization*, pages 1–40. Springer, 1993.

[16] F. J. Post, T. van Walsum, and F. H. Post. Iconic techniques for feature visualization. In *Proceedings of IEEE Visualization '95*, pages 288–295, 1995.

[17] M. Rumpf and R.-T. Happe. Characterizing gobal features of simulation data by selected local icons. In *Virtual Environments and Scientific Visualization '96*. Springer Wien, 1996.