Computing the tension parameters for the composite rational Bezier curves

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Abstract: A system for computing the tension parameters at the join point of two rational Bezier curves is presented. The curves belong to the C^1/C^2 class.

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1. Introduction

Let $V_0, \ldots, V_{2k}, k = 2, 3, \ldots$, with the coordinates $V_i = (V_i^x, V_i^y, V_i^z)$ be the given points of the Euclidean space E_3 and $w_0, \ldots, w_{2k}, k = 2, 3, \ldots$, be their weights. By these weights (positive real numbers) are created in the Euclidean space E_4 the points $\overline{V_i} = (w_i V_i^x, w_i V_i^y, w_i V_i^z, w_i), i = 0, \ldots, 2k, k = 2, 3, \ldots$.

A single Bezier curve of the degree $2k, k = 2, 3, \ldots$, in E_4 is defined by

$$\mathrm{R}(t) = \sum_{i=0}^{2k} \overline{V_i} B_{2k,i}(t) \qquad t \in \langle 0, 1 \rangle$$

where $B_{2k,i}(t)$ are Bernstein polynomials [ZATKO]. The projection with the centre $O(0,0,0,0)$ of the Bezier curve $\mathrm{R}(t)$ into the hyperplane $\rho \equiv x_4 - 1 = 0$ is the rational Bezier curve with the control polygon in the vertices $V_0, \ldots, V_{2k}, k = 2, 3, \ldots$, and has an analytical expression

$$\mathrm{r}(t) = \frac{\sum_{i=0}^{2k} w_i V_i B_{2k,i}(t)}{\sum_{i=0}^{2k} w_i B_{2k,i}(t)} \qquad t \in \langle 0, 1 \rangle \tag{1}$$

Let us denote $W(w_0, w_1, \ldots, w_{2k-1}, w_{2k})$ to be a weight-vector. For the next considerations we take the weights $w_0 = w_{2k} = 1, w_i > 0, i = 1, \ldots, 2k-1$, and we concentrate on the following substitutions:

- $w_i B_{2k,i}(t) = \alpha_i t^i (1-t)^{2k-i}$, where $\alpha_i = w_i \begin{pmatrix} 2k \\ i \end{pmatrix}$ and we call α_i the tension parameter of the point V_i [HOSCH], [JOSC]

- the denominator in (1) depending on the weight-vector $(1, w_1, \ldots, w_{2k-1}, 1)$

has the form

$$\sum_{i=0}^{2k} w_i B_{2k,i}(t) = (1-t)^{2k} + \sum_{i=1}^{2k-1} \alpha_i t^i (1-t)^{2k-1} + t^{2k} = \omega(\alpha, t)$$

- using the identity

$$(1-t)^{2k+1} + t^{2k+1} = (1-t)^{2k} + \sum_{i=1}^{2k-1} (-1)^i t^i (1-t)^{2k-i} + t^{2k}$$

the function $\omega(\alpha, t)$ can be rewritten as

$$\omega(\alpha, t) = (1-t)^{2k+1} + \sum_{i=1}^{2k-1} (\alpha_i - (-1)^i) t^i (1-t)^{2k-i} + t^{2k+1}.$$

Applying these substitutions into (1) the analytic expression of the rational Bezier curve has the form:

$$\mathrm{r}(t) = \mathrm{r}(\alpha, t) = \sum_{i=0}^{2k} V_i RB_{2k,i}(\alpha, t) \tag{2}$$

where $RB_{2k,i}(\alpha, t)$ are the rational functions with the tension parameters $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{2k-1})$ and their formulas are:

$$RB_{2k,0}(\alpha, t) = \frac{(1-t)^{2k+1}}{\omega(\alpha, t)}$$

$$RB_{2k,i}(\alpha, t) = \frac{(\alpha_i - (-1)^i) t^i (1-t)^{2k-i}}{\omega(\alpha, t)} \qquad i = 1, \dots, 2k-1$$

$$RB_{2k,2k}(\alpha, t) = \frac{t^{2k+1}}{\omega(\alpha, t)}$$

We note that the functions $RB_{2k,i}(\alpha, t)$:
- are nonnegative over the interval $\langle 0, 1 \rangle$
- form a partition of unity
- depend on the tension parameters.

Now we deal with the geometric properties of the rational Bezier curve expressed as the linear combination between rational functions $RB_{2k,i}(\alpha, t)$ and the vertices $V_0, \dots, V_{2k}, k = 2, 3, \dots$, of the control polygon.

The curve
- interpolates the endpoints

$$\mathrm{r}(\alpha, 0) = V_0, \qquad \mathrm{r}(\alpha, 1) = V_{2k}$$

- the first derivate in the endpoints

$$\mathrm{r'}(\alpha, 0) = (\alpha_1 + 1)(V_1 - V_0)$$

$$\mathrm{r'}(\alpha, 1) = (\alpha_{2k-1} + 1)(V_{2k} - V_{2k-1})$$

- the curvature in the endpoints

$$\mathrm{k}(\alpha, 0) = \frac{2(\alpha_2 - 1)|V_1 - V_2||\sin\gamma_1|}{(\alpha_1 + 1)^2|V_1 - V_0|^2}$$

$$\mathrm{k}(\alpha, 1) = \frac{2(\alpha_{2k-2} - 1)|V_{2k-2} - V_{2k-1}||\sin\gamma_{2k-1}|}{(\alpha_{2k-1} + 1)^2|V_{2k-1} - V_{2k}|^2}$$

$$\gamma_i = \angle V_{i-1}V_iV_{i+1}, i = 1, 2k - 1$$

Now we concentrate on applying these results to join two control polygons and thus to obtain C^2-continous curve.

2. The composite Bezier curve and the tension parameters

Let the control polygon CP$[V_i, i = 0, \ldots, 2k]$ consist of the points $V_0, \ldots, V_{2k}$, $k = 2, 3, \ldots$. By the expression (2) the rational Bezier curve is described, which reflects the shape of the control polygon.

The shape modifications of this curve can be arranged by

- changing the value of the weights (for special placing of the control vertices and the selection of the weights the symmetric Bezier curve can be obtained [KUD])

- the additional control points (the multiple control points).

In some cases, above all when the number of the control vertices is over seven (for $k \geq 3$), the curve follows the control polygon but not as closely as we desire. Because of increasing the number of the control points the degree of the curve increases too and the curve is pulled away from the control points (except the endpoints).

To prevent this situation, we suggest to join the single rational Bezier curves and to create the rational Bezier spline.

In this process, let CP$[P_i, i = 0, \ldots, 2l, l = 2, 3, \ldots]$ be a control polygon and r(α^P, t) its rational Bezier curve (α^P -denotes the dependence of the curve on the tension parameters belonging to the control vertices $P_i, i = 1, \ldots, 2l - 1$, punctually $\alpha^P = (\alpha_1^P, \ldots, \alpha_{2l-1}^P), \alpha_i^P = \alpha_i^{P_i}$ and contains the value of the tension parameter of each control vertex).

Let us join this curve r(α^P, t) with a curve r(α^Q, t) to be fulfiled the conditions of C^0, C^1, C^2 continuity at the join point of both segments. The vertices of a new control polygon CP$[Q_i, i = 0, \ldots, 2m, m = 2, 3, \ldots]$ are denoted as $Q_0, \ldots, Q_{2m}$.

Now we write the constraints of the continuity and we deal with computing of the tension parameters α^Q for the vertices $Q_i, i = 1, 2$:

Continuity:

C^0 : $\mathrm{r}(\alpha^P, 1) = \mathrm{r}(\alpha^Q, 0)$ thus $P_{2l} = Q_0$

$$C^1 : \mathrm{r'}(\alpha^P, 1) = \mathrm{r'}(\alpha^Q, 0) \text{ thus } (P_{2l-1}Q_1P_{2l}) = -1 \text{ and } \alpha^P_{2l-1} = \alpha^Q_1 \tag{3}$$

C^2 : $\mathrm{r''}(\alpha^P, 1) = \mathrm{r''}(\alpha^Q, 0)$ and the tension parameter can be derived from the curvature in the endpoints: $\mathrm{k}(\alpha^P, 1) = \mathrm{k}(\alpha^Q, 0)$ thus

$$\alpha^Q_2 = 1 + \frac{(\alpha_{2l-2} - 1)|P_{2l-2} - P_{2l-1}|| \sin \gamma^P_{2l-1}|}{|Q_1 - Q_2|| \sin \gamma^Q_1|} \tag{4}$$

3. The control polygon and its curves with the tension parameters

The above results can be applied as a useful tool to create the rational Bezier spline of the class C^2. For this process, we suggest to work with the control polygon $\mathrm{CP}[V_i, i = 0, \ldots, r]$ and we allow the Bezier spline to touch the control polygon (except the endpoints V_0, V_r). According to these conditions, we can realize the next steps:

-select an edge of the control polygon $\mathrm{CP}[V_i, i = 0, \ldots, r]$, that the join point of two curve segments coincides with, for example the edge $V_{2l-1}V_{2l}, 1 \leq l \leq r-2$

-create two control polygons using the vertices of the original control polygon and call them as

$\mathrm{CP}[P_i, i = 0, \ldots, 2l]$: $P_i = V_i$
$P_{2l} = \frac{1}{2}(V_{2l-1} + V_{2l})$

and

$\mathrm{CP}[Q_s, s = 0, \ldots, r - 2l]$: $Q_0 = P_{2l}$
$Q_{s+1} = V_{2l+s}$.

Remark: Before rearranging the vertices we have supposed the number of the control vertices V_i is odd. By dividing we have obtained two subpolygons and one of them, the subpolygon $\mathrm{CP}[Q_s, s = 0, \ldots, r - 2l]$, has even number of the control vertices. The described construction of the rational Bezier curve needs odd number of the control vertices thus we add one control vertex (the multiple endpoint).

-apply the expressions (3), (4) to suggest the Bezier spline by composing the rational Bezier curves with C^2-continuity. After rewriting these formulas for the vertices V_i of the original control polygon, they have the form:

$$\frac{1}{2}(V_{2l-1} + V_{2l}) = P_{2l} = Q_0 \tag{5}$$

$$\alpha^V_{2l-1} = \alpha^V_{2l} \tag{6}$$

$$\alpha^V_{2l+1} = 1 + \frac{(\alpha_{2l-2} - 1)|V_{2l-2} - V_{2l-1}||\sin\gamma^V_{2l-1}|}{|V_{2l} - V_{2l+1}||\sin\gamma^V_{2l}|} \tag{7}$$

The above formulas for computing the tension parameters guarantee the construction of the curve with C^2-continuity at the join points.

For the original control polygon, we can suggest many curves, according to the demands how close a curve can come to the control polygon and how many edges of the control polygon a curve can touch.

4. Conclusion

The described method offers for one control polygon to create a family of the curves with C^2 continuity at the join points. The curves follow the control polygon as closely as desired without moving original control vertices.

In many applications, we are forced to use the curves with corners. This problem can be solved with C^0 or C^1 continuity at the join points. It means to accept the formulas (5) or (6). This shape control allows to create the curves from the zones of the rational Bezier segment with C^0, C^1 or C^2 continuity at the join points.

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