Approximation and Smoothing with Spline curves using Multicriteria Optimization Techniques

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Abstract

The design of freeform curves and surfaces is one of the major topics of CAD/CAM. Often data points detected by a scanning process have to be interpolated or approximated. For aesthetic and technical requirements they have to fulfill quality criteria. Quality diminishing properties like dents or local violating of convexity must be eliminated while approximation conditions must hold at the same time. We combine least square fitting of Bézier spline curves with criteria, based on variational design principle and convexity analysis in a non linear multicriteria optimimization problem. For an automatic approximation and smoothing process the weighted sum multicriteria optimization approach is used to detect the efficient solutions.

1 Introduction

The design of freeform curves and surfaces is one of the major topics of CAD/CAM. Often data points detected by a scanning process have to be interpolated or approximated by curves or surfaces. The resulting curves or surfaces are desired to be also technical smooth. For that purpose W. Hohenberger and T. Reuding [7] combined minimization of the rate of curvature change and minimization of weight differences of NURBS curves to minimize the curvature and shape variation of a initial NURBS curve or surface approximating the given data points.

Least square fitting is a different powerfull approach for approximating given data points itself via parametric curves. However, inaccurate measurements usually cause dents and wavy shape of the resulting curve which means inflexion points of its curvature. Aesthetic and technical requirements force us to add smoothing and convexity (respectively concavity) criteria supplementary to the approximation criteria. In the work [6] of H. Hagen and P. Santarelli the minimization of the integral $\int \alpha \|X''(t)\|^2 + \beta \|X'''(t)\|^2 dt$, together with a least square constraint is used. In this approach we use criteria based on variational design principle and convexity criteria additional to least square fitting. The curvature is fundamental for such criteria. As our investigation deals with planar parametric curves, signed curvature is a promising basis to obtain convexity criteria.

The combination of these criteria leads up to a non linear multicriteria optimization problem. So we receive not only one optimal solution but several *efficient* solutions which fulfill the constraints and make all objective as small as possible. For an automatic approximation and smoothing process the weighted sum multicriteria optimization approach is used to detect the efficient solutions. As we achieve one efficient solution at a time applying weighting sums we have to search for a suitable weighting.

2 Mathematical fundamentals

2.1 Multicriteria optimization problems

In problems with multiple objectives a so-called *pareto optimal* or *efficient solution* has to fulfill the constraints and make the components of the objective function vector as small as possible. A *multicriteria optimization problem* or *multiple objectives problem* is given as

min { f(x)|h(x)=0, g(x) $\leq$b, b$\epsilon\mathrm{I\!R}^n$, x$\epsilon\mathrm{I\!R}^n_+$}

where f = $(f_1, \ldots, f_p)$ is a tuple of objective functions named criteria vector, p, nϵIN and h and g are (not necessary linear) restriction functions.

The multicriteria optimization problem differs from single criterion only by its explicit consideration of multiple criteria.

However this is an important difference, as it puts the designer in a position of providing a set of good, alternative solutions rather than a single *optimal* solution.

By the *weighted sums* approach the k weighted objectives are summed to form one (weighted sums) objective function. The resulting optimization problem is called *weighted sum multicriteria optimization problem.* Without loss of generality it can be assumed that the weighting vector is normalized such that the sums of its elements equals to one. Each optimal solution of such a weighted sum multicriteria optimization problem is an efficient solution of the corresponding multicriteria optimization problem. So one can variate the weights and get a set of efficient solutions through weighted sums. In practice interactive procedures have proven to be most effective in searching for a final solution.

2.2 Curvatures of planar curves

The curvature of parametric curves is essential for smoothing criterias. In the case of planar parametric curves we differ between *signed* and *unsigned curvature.* The definition of unsigned curvature corresponds to the common curvature definition. Let φ be an at least twice differentiable parametric curve from [a,b]$\subseteq$IR into IR^2. The *unsigned curvature* of φ at tϵ[a,b] is given by

$k_{unsigned}(t) = \frac{||det(\dot{\varphi}(t),\ddot{\varphi}(t))||}{||(\dot{\varphi}(t))||^3}$,

wheras the *signed curvature* of φ at tϵ[a,b] is given by

$k_{signed}(t) = \frac{det(\dot{\varphi}(t),\ddot{\varphi}(t))}{||\dot{\varphi}(t)||^3}$.

The dots denote the differentiation with respect to t. To accentuate the correlation to the definition of curvature in n–dimensional case we will write curvature instead of unsigned curvature and denote it with k.

The signed curvature distinguishes between convex and concave parts of the curve and will be used for a criteria to get either a locally convex or a locally concave curve when minimized.

3 Formulation of the multicritera optimization problem

For approximation of given data points we employ planar quintic Bézier spline curves on the parameter intervall [0,1]. We consider planar Bézier spline curves of degree n as segmented curves of degree n. Let $(t_0 = 0, \ldots, t_m = 1)$ be a parametrization of a curve B:$[t_0, t_m]$:$\rightarrow$IR^2 of degree n. The bézier segments $B_k (k = 0, \ldots, m-1; m\epsilon\mathbb{N})$ of B are

$B_k(t) = \sum_{i=0}^{n} b_{nk+i} B_i^n(\frac{t-t_k}{t_{k+1}-t_k})\ (t\epsilon[t_k,t_{k+1}])$,

where $b_i\epsilon$IR^2 and $B_i^n(t) = \begin{pmatrix} n \\ i \end{pmatrix} (1-t)^{n-i}t^i$ are Bernstein polynomials.

We manipulate the curve by moving control points b_i with respect to our approximation and smoothing criteria. As mentioned above smoothing and convexity (respectively concavity) criteria have to be added by aesthetic demands. We receive the following non linear constrained multicriteria optimization problem.

min $\{c_i(\text{b})|$ Ab=0, $b_j{\geq}0\}$

where c_i are approximation or smoothness criteria, A is the restriction matrix and b_j=$(b_{j0}, b_{j1})^T$ are the control points, b=$(b_{0_{(1)}}, .., b_{mn_{(2)}})^T$, i =1,...,p; j=0,...,mn; i, j,p $\epsilon\mathbb{N}$.

We combine the objective function as a weighted sum to find the most favored efficient solution. Because we regard a multicriteria optimization problem there will be more than one efficient solution.

For approximation of data points the minimum square method is an approved approach and therefore is used as first objective function:

$c_1(b) = \sum_{i=0}^{n_1} (B(\tilde{t_{i_1}}) - z_i)^2$

(z_i points to be approximated, $\tilde{t}_{0_1}$=0, $\tilde{t}_{n_1}$=1, $\tilde{t}_{i_1-1} \leq \tilde{t}_{i_1} \leq \tilde{t}_{i_1+1}$,

$i = 1, \ldots, n_1 - 1; n_1 \in \mathbb{N}$)

As smoothing criteria we adopt the linearization of strain energy $\int_0^1 k^2(t) \mid \dot{B}(t) \mid dt$ and use the trapezial rule to get the second criterion

$c_2(b) =$

h[0.5 $k^2(\tilde{t}_{0_2}) \parallel \dot{B}(\tilde{t_{0_2}}) \parallel + \sum_{j=1}^{n_2-1} k^2(\tilde{t}_{j_2}) \parallel \dot{B}(\tilde{t_{j_2}}) \parallel + 0.5 k^2(\tilde{t}_{n_2}) \parallel \dot{B}(\tilde{t_{n_2}}) \parallel$],

where h=$\frac{1}{n_2}$ is the step length, $\tilde{t}_{1_2}$=0, $\tilde{t}_{n_2}$=1, $\tilde{t}_{j_2} = \tilde{t}_{j1_2} + h, j_2 = 1, \ldots, n_2 - 1;$ $n_2 \in \mathbb{N}$.

We avoid the approximation of the curvature with the norm of the second derivative in order to be independent from the arc length parametrization.

We stated in the previous section that the signed curvature is an accurate detector for convexity or concavity. It is convex if it is positive and concave if it is negative and a change of its sign marks an alternation of a convex part in a concave part. Based on this property of signed curvature we introduce the following function to avoid inflexion points of curvature.

$$K = \frac{\int_0^1 sign(det(\dot{B}(\tilde{t}_{i_3}),\ddot{B}(\tilde{t}_{i_3})))det(\dot{B}(\tilde{t}_{i_3}),\ddot{B}(\tilde{t}_{i_3}))^2+1}{\int_0^1 det(\dot{B}(\tilde{t}_{i_3}),\ddot{B}(\tilde{t}_{i_3}))^2+1}$$

where $\tilde{t}_{1_3}$=0, $\tilde{t}_{n_3}$=1, $\tilde{t}_{j_3-1} \leq \tilde{t}_{j_3} \leq \tilde{t}_{j_3+1}, j_3 = 1, \ldots, n_3 - 1$; $n_3 \in \mathbb{N}$.

K equals to one if and only if the sign of the curvature does not change. Therefore the function $c_2(b)$ =1-$K^2(b)$ is minimized exactly when the sign of the curvature does not change and inflexion points do not exist.

We use the discretization of this as third criterion

$$c_3(B) = 1 - (\frac{\sum_{i=0}^{n_3} sign(det(\dot{B}(\tilde{t}_{i_3}),\ddot{B}(\tilde{t}_{i_3})))det(\dot{B}(\tilde{t}_{i_3}),\ddot{B}(\tilde{t}_{i_3}))^2+1}{\sum_{i=1}^{n_3} det(\dot{B}(\tilde{t}_{i_3}),\ddot{B}(\tilde{t}_{i_3}))^2+1}))^2$$

where $\tilde{t}_{1_3}$=0, $\tilde{t}_{n_3}$=1, $\tilde{t}_{j_3-1} \leq \tilde{t}_{j_3} \leq \tilde{t}_{j_3+1}, j_3 = 1, \ldots, n_3 - 1$; $n_3 \in \mathbb{N}$.

This criterion works sufficiently in practice. We give an example below.

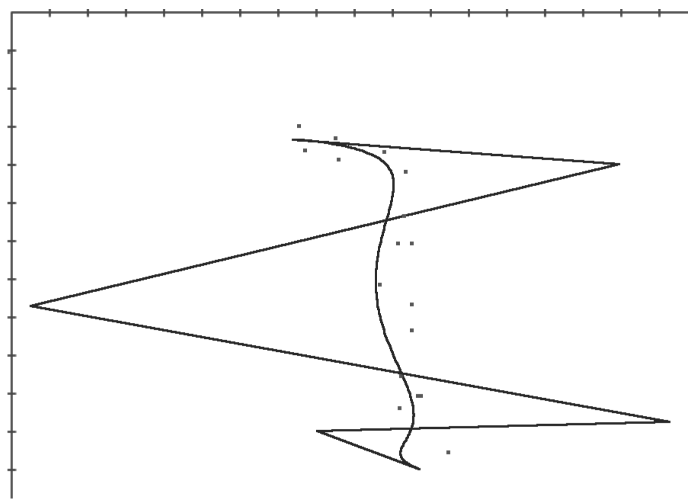

Figure 1: The original curve, its control polygon and approximation points.

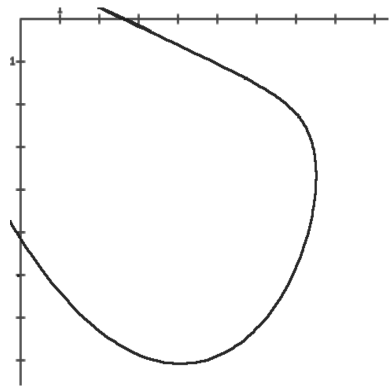

Figure 2: The resultating curve without its control polygon after optimization with criteria c3 (n_3=20).

4 Approximation and smoothing process

We use a gradient search algorithm solving the constructed optimization problem. The gradient of the regarded weighted criteria vector

$$\mathrm{grad}\Big(\sum_{j=1}^{p} w_j c_j(b)\Big) = \Big(\sum_{j=1}^{p} w_j \frac{\partial}{\partial b_{0_1}} c_j(b), \ldots, \sum_{j=1}^{p} w_j \frac{\partial}{\partial b_{nm_2}} c_j(b)\Big)$$

is itself a weighted sum of gradients of each criteria. The weights $w_j (j = 1, \ldots, p)$ are non negative real numbers which sums to one.

We develop the partial derivatives for the more general case of planar Bézier spline curves of degree n ($n \epsilon \mathbb{N}$) on the parameter intervall [a,b]$\epsilon \mathbb{R}$. For computation of the current gradients the partial derivatives of the following functions are important:

A(t)= $det((\dot{B}_k(t), \ddot{B}_k(t))$

$= det(\frac{n!}{(n-1)!}\frac{1}{(b-a)}\sum_{i=0}^{n-1}\Delta b_{kn+i}B_i^{n-1}(t), \frac{n!}{(n-2)!}\frac{1}{(b-a)^2}\sum_{j=0}^{n-2}\Delta b_{kn+j}B_j^{n-2}(t))$

$=nn(n-1)\frac{1}{(b-a)^3}[\sum_{i=0}^{n-1}\sum_{j=0}^{n-2}\Delta b_{kn+i_{(1)}}\Delta b_{kn+j_{(2)}}B_i^{n-1}(t)B_j^{n-2}(t)$

$-\sum_{i=0}^{n-1}\sum_{j=0}^{n-2}\triangle b_{kn+i_{(2)}}\Delta b_{kn+j_{(1)}}B_i^{n-1}(t)B_j^{n-2}(t)]$

In the following we denote A(t) with A and write A_i instead of A($\tilde{t_i}$) for an parameter value $\tilde{t_i}$.

For the partial derivatives in the first component of b_l follows:

($l = 0,\ldots$,n; n is the degree of the Bézier spline curve, for j < 1 and j > n, B_j^n is set equal to 0)

$A^{l_1}=\frac{\partial}{\partial b_{kn+l_{(1)}}}$A

$=nn(n-1)\frac{1}{(b-a)^3}[\sum_{j=0}^{n-2}\triangle b_{kn+j_{(2)}}B_{l-1}^{n-1}(t)B_j^{n-2}(t) - \sum_{j=0}^{n-2}\triangle b_{kn+j_{(2)}}B_l^{n-1}(t)B_j^{n-2}(t)$

$-\sum_{i=0}^{n-1}\triangle b_{kn+i_{(2)}}\ B_i^{n-1}(t)B_{l-1}^{n-2}(t) + \sum_{i=0}^{n-1}\triangle b_{kn+i_{(2)}}\ B_i^{n-1}(t)B_l^{n-2}(t)]$

$= nn(n-1)\frac{1}{(b-a)^3}[\sum_{j=0}^{n-2}\triangle b_{kn+j_{(2)}}(B_{l-1}^{n-1}(t)B_j^{n-2}(t)-B_l^{n-1}(t)B_j^{n-2}(t)-B_j^{n-1}(t)B_{l-1}^{n-2}(t)+$
$B_j^{n-1}(t)B_l^{n-2}(t)) + \triangle b_{kn+(n-1)_{(2)}}(-B_{n-1}^{n-1}(t)B_{l-1}^{n-2}(t) + B_{n-1}^{n-1}(t)B_l^{n-2}(t))]$

Analogous we receive:

$A^{l_2}=\frac{\partial}{\partial b_{kn+l_{(2)}}}$ A

$= nn(n-1)\frac{1}{(b-a)^3}[-\sum_{i=0}^{n-2}\triangle b_{kn+i_{(1)}}(B_{l-1}^{n-1}(t)B_i^{n-2}(t)-B_l^{n-1}(t)B_i^{n-2}-B_i^{n-1}B_{l-1}^{n-2}(t)$

$+ \mathrm{B}_i^{n-1}(t)B_l^{n-2}(t)) + \triangle b_{kn+(n-2)_{(1)}}(B_{n-1}^{n-1}(t)B_{l-1}^{n-2}(t) - B_{n-1}^{n-1}(t)B_l^{n-2}(t))]$

The following equations hold for m $\in$ {1, 2 }.

$B^{l_m} = \frac{\partial}{\partial b_{kn+l_{(m)}}}||\ \dot{B}_k(t)\ ||^3$

$= \frac{3}{2}\ ||\ \dot{B}_k(t)\ ||\ \frac{\partial}{\partial b_{kn+l_{(m)}}}\sum_{i=1}^{m}(\dot{B}_k(t))_i^2$

$= 3\ ||\ \dot{B}_k(t)\ ||\ \frac{n}{(b-a)}(B_{l-1}^{n-1}(t) - B_l^{n-1}(t))$

The partial derivatives of the curvature are given as follows:

$k^{l^m} = \frac{\partial}{\partial b_{kn+l_{(m)}}}k(t) = \frac{sign(A)\ A^{l_m}B-AB^{l_m}}{B^2}$

It follows:

$$\frac{\partial}{\partial b_{kn+l_{(m)}}} c_1(b) = 2\sum_{i=0}^{n_2}(\sum_{j=0}^{n} b_{kn+j_m} B_j^n(\tilde{t}_{i_1}) - P_{i_m}) B_l^n(\tilde{t}_{i_1})$$

$$\frac{\partial}{\partial b_{kn+l_{(m)}}} c_2(b)$$

$$=h[k^{l_{(m)}}(\tilde{t}_{0_2})k(\tilde{t}_{0_2}) \ \|\ \dot{B}_k(\tilde{t}_{0_2})\ \| + 0.5k^2(\tilde{t}_{0_2})\frac{\partial}{\partial bkn+_{l_{(m)}}}\ \|\ \dot{B}_k(\tilde{t}_{0_2})\ \|$$

$$+ 2\sum_{j=1}^{n_3-1}(k^{l_m}(\tilde{t}_{j_2})k(\tilde{t}_{j_2})\ \|\ \dot{B}_k(\tilde{t}_{j_2})\ \| + k^2(\tilde{t}_{j_2})\frac{\partial}{\partial b_{kn+l_{(m)}}}\ \|\ \dot{B}_k(\tilde{t}_{j_2})\ \|) + k^{l_m}(\tilde{t}_{n_2})k(\tilde{t}_{n_2})\ \|\ \dot{B}_k(\tilde{t}_{n_2})\ \| + 0.5k^2(\tilde{t}_{n_2})\frac{\partial}{\partial b_{kn+l_{(m)}}}\ \|\ \dot{B}_k(\tilde{t}_{n_2})\ \|],$$

$$\frac{\partial}{\partial b_{l_{(m)}}} c_3(b)$$

$$=-2\left(\frac{\sum_{i=0}^{n_3}(A_{i_3}|A_{i_3}|)+1}{\sum_{i=0}^{n_3} A_{i_3}^2+1}\right)\left(\frac{2\sum_{i=0}^{n_3}(A_{i_3}^{l_m}|A_{i_3}|)(\sum_{i=0}^{n_3} A_{i_3}^2+1)-(\sum_{i=0}^{n_3} A_{i_3}|A_{i_3}|+1)2\sum_{i=0}^{n_3}(A_{i_3}^{l_m}A_{i_3})}{(\sum_{i=0}^{n_3} A_{i_3}^2+1)^2}\right.$$

$$=-4\left(\frac{\sum_{i=0}^{n_3}(A_{i_3}|A_{i_3}|)+1}{(\sum_{i=0}^{n_3} A_{i_3}^2+1)^2}\right)\left(\sum_{i=0}^{n_3}(A_{i_3}^{l_m}\ |\ A_{i_3}\ |) - \left(\frac{\sum_{i=0}^{n_3}(A_{i_3}|A_{i_3}|)+1}{\sum_{i=0}^{n_3} A_{i_3}^2+1}\right)\sum_{i=0}^{n_3}(A_{i_3}A_{i_3}^{l_m})\right),$$

Gradient search algorithms require a continous differentiable objective function. These requirements together with the employment of curvature in our objective functions force the curve to have a continous differentiable curvature. We solve that problem adding C^3–restrictions and obey a feasable direction strategy. C^3–continuity guarantees continuously differentiable curvature all over the curve.

The restrictions of C^3–continuity leads to the linear system which is expressed by the equations I.-III. for $k = 1, \ldots, m-1$:

I. $-\frac{1}{t_k-t_{k-1}}b_{nk-1}+(\frac{1}{t_k-t_{k-1}} + \frac{1}{t_{k+1}-t_k})b_{nk}\text{-}\frac{1}{t_{k+1}-t_k}b_{kn+1} = 0$

II. $\frac{1}{(t_k-t_{k-1})^2}b_{kn-2}$ - $\frac{2}{(t_k-t_{k-1})^2}b_{kn-1} + (\frac{1}{(t_k-t_{k-1})^2}\text{-}\ \frac{1}{(t_{k+1}-t_k)^2})b_{kn} + \frac{2}{(t_{k+1}-t_k)^2}b_{kn+1}$

-$\frac{1}{(t_{k+1}-t_k)^2}b_{kn+2}= 0$

III. -$\frac{1}{(t_k-t_{k-1})^3}b_{kn-3}+\ \frac{3}{(t_k-t_{k-1})^3}b_{kn-2}\text{-}\ \frac{3}{(t_k-t_{k-1})^3}b_{kn-1} + (\frac{1}{(t_k-t_{k-1})^3}+\frac{1}{(t_{k+1}-t_k)^3})b_{kn} - \frac{3}{(t_{k+1}-t_k)^3}b_{kn+1}$
$+\ \frac{3}{(t_{k+1}-t_k)^3}b_{kn+2}$ - $\frac{1}{(t_{k+1}-t_k)^3}b_{kn+3}= 0$

The restriction matrix is set as follows:

$$A = \begin{pmatrix} a_1^T \\ \vdots \\ a_{3m-3}^T \end{pmatrix} \text{ where } a_l = \begin{pmatrix} a_l^{(0)} \\ \vdots \\ a_l^{(mn-1)} \end{pmatrix} \text{ is set for } k = 1, \ldots, m-1:$$

$a_{k-1}^{(nk-1)} = -\frac{1}{t_k - t_{k-1}};$

$a_{k-1}^{(nk)} = \frac{1}{t_k - t_{k-1}} + \frac{1}{t_{k+1} - t_k};$

$a_{k-1}^{(nk+1)} = -\frac{1}{t_{k+1} - t_k};$

$a_{k-1}^{(j)} = 0 \ (0 \leq j \leq mn, j \notin \{nk-1, nk, nk+1\};)$

$a_{m+k}^{(kn-2)} = \frac{1}{(t_k - t_{k-1})^2} \ ;$

$a_{m+k}^{(kn-1)} = - \frac{2}{(t_k - t_{k-1})^2};$

$a_{m+k}^{(kn)} = \frac{1}{(t_k - t_{k-1})^2} - \frac{1}{(t_{k+1} - t_k)^2};$

$a_{m+k}^{(kn+1)} = +(\frac{2}{(t_{k+1} - t_k)^2};$

$a_{m+k}^{(kn+2)} = -\frac{1}{(t_{k+1} - t_k)^2};$

$a_{m+k}^{(j)} = 0 \ (0 \leq j \leq mn, j \notin \{nk-2, nk-1, nk, nk+1, nk+2\});$

$a_{k+2m-2}^{(kn-3)} = -\frac{1}{(t_k - t_{k-1})^3};$

$a_{k+2m-2}^{(kn-2)} = \frac{3}{(t_k - t_{k-1})^3};$

$a_{k+2m-2}^{(kn-1)} = - \frac{3}{(t_k - t_{k-1})^3};$

$a_{k+2m-2}^{(kn)} = \left(\frac{1}{(t_k - t_{k-1})^3} + \frac{1}{(t_{k+1} - t_k)^3}\right);$

$a_{k+2m-2}^{(kn+1)} = -\frac{3}{(t_{k+1}) - t_k)^3};$

$a_{k+2m-2}^{(kn+2)} = \frac{3}{(t_{k+1} - t_k)^3};$

$a_{k+2m-2}^{(kn+3)} = - \frac{1}{(t_{k+1} - t_k)^3};$

$a_{k+2m-2}^{(j)} = 0 \ .$

5 Application

For approximation of given data points we employ planar quintic Bézier spline curves on the parameter intervall [0,1]. Since Bézier segments are $C^{\propto}$–differentiable we can drop the C^3–restrictions and get the following optimization problem, given as

$\min \ \{c_i(b) | b \geq 0, i = 1, .., 3\},$

which allows us to solve it with an usual conjugate gradient algorithm.

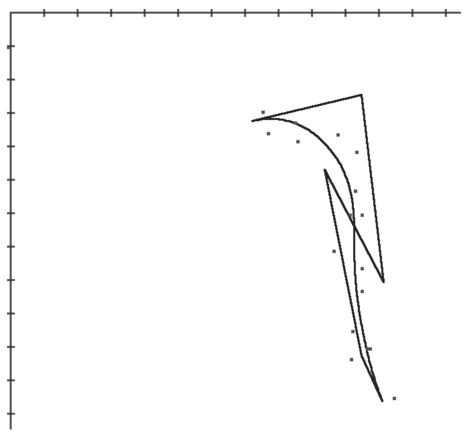

Figure3: The resultating curve, its control polygon and the approximation points after optimization with criteria 1, 2 and 3 ($n_1 = 17$,$n_2 = n_3$=20). The weightings are $w_1 = 0.13333333$, $w_2 = 0.03333333$ and $w_3 = 0.83333333$.The initial curve is given in figure1.

For Bézier splines we have to solve the restricted problem. We use the Reduced Gradient Method of WOLFE.

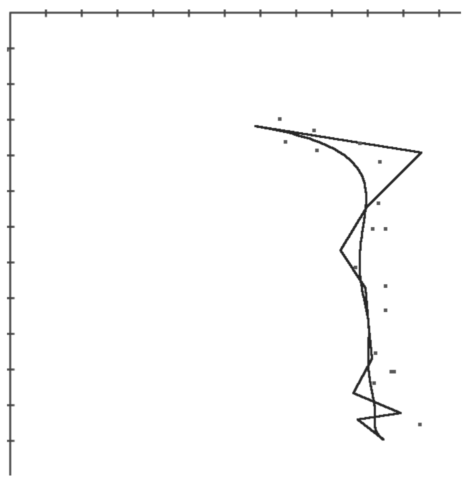

Figure3: The resultating curve, its control polygon and the approximation points after optimization with criteria 1, 2 and 3 ($n_1 = 17$, $n_2 = n_3$=20). The weightings are $w_1 = 0$, $w_2 = 0.074074074$ and $w_3 = 0.92593$. The initial curve is given by the curve in figure1 segmented at the parameter value 0.5. One can see the smoothing effect.

Conclusion

We presented in this paper a method for approximation and smoothing of planar curves using multicriteria optimization techniques. Smoothing and convexity (respectively concavity) criteria were added supplementary to the approximation criteria.

Since least square fitting possibly gives good results approximating point data which are already smooth, whereas inaccurate measurements usually cause dents and wavy shape of the resulting curve, for aesthetic and technical requirements it is of considerable interest to look at the effort of smoothing and rounding effect besides the approximation.

The results in the previous section show distinctly the smoothing effect of the strain energy criteria in combination with the rounding effect of the convexity criteria contemprorary considering the approximation aim when choosing an appropriate weight in the weighting sum approach. This effect is strenghtened by the segmentation shown in *Figure3*.

Further research concerned with finding suitable segmentations and extending this method to tensor product Bézier surfaces will be done.

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