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Abstract

Spline curves and spline surfaces are a powerful modelling tool. They have many useful properties, but using spline techniques to model topologically arbitrary surface is hard. If the spline surface is created over a rectangular control grid, it can be difficult modelling the surface, which topology can be different from the topology of the control grid. The goal of this paper is to overcome this restriction using binding sides of the control grid. We will show, how we can model some basic topological objects (cylinder, Moebius strip, torus, sphere, cone, Klein bottle) with spline surfaces. We also describe some properties of these surfaces and practical tips for application.

1. Introduction

We can define a spline or a spline surface as piecewise polynomial point function depending on 1 or 2 real variables respectively [BARS88]. A spline segment can be expressed as a linear combination of control points with blending functions. The most frequently used splines are piecewise cubic functions.

For the sake of simplicity we will deal with uniform cubic B-splines only.

For a large group of splines (B-splines, β-splines, cardinal splines, NURBS, etc.) we can define the spline segment:

$$S_k(V_k, V_{k+1}, V_{k+2}, V_{k+3}) = \sum_{i=0}^{3} f_i(t).V_{k+i} \qquad (1)$$

S_k is a cubic spline segment, where $f_i(t)$ are blending functions and V_i are control points.

If we have N+1 control points $V_0...V_N$, then the spline is defined:

$$S = \sum_{k=0}^{N-3} S_k(V_k, V_{k+1}, V_{k+2}, V_{k+3}) \qquad (2)$$

As known, **two spline segments S_k(A,B,C,D) and S_l(X,Y,Z,W) are smoothly connected** if: **B = X, C = Y, D = Z** (3)

2. B-*spline surfaces*

The blending functions for the uniform cubic B-spline are given by the following formulas:

$f_0(u) = (-u^3+3u^2-3u+1)/6$,

$f_1(u) = (3u^3-6u^2+4)/6$, (4)

$f_2(u) = (-3u^3+3u^2+3u+1)/6$,

$f_3(u) = (u^3)/6$, where $u \in [0,1]$.

A spline curve segment can expressed in a matrix form:

$$S_k(u) = \mathbf{u.M.V}^T = (1/6).(u^3, u^2, u, 1).\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix} V_k \\ V_{k+1} \\ V_{k+2} \\ V_{k+3} \end{pmatrix} \qquad (5)$$

A spline surface segment can expressed in a matrix form:

$$S_{k,j}(u,v)= \mathbf{u.M.}\begin{pmatrix} V_{k,j} & V_{k+1,j} & V_{k+2,j} & V_{k+3,j} \\ V_{k,j+1} & V_{k+1,j+1} & V_{k+2,j+1} & V_{k+3,j+1} \\ V_{k,j+2} & V_{k+1,j+2} & V_{k+2,j+2} & V_{k+3,j+2} \\ V_{k,j+3} & V_{k+1,j+3} & V_{k+2,j+3} & V_{k+3,j+3} \end{pmatrix}\mathbf{.M^T.v^t} \qquad (6)$$

A B-spline surface is defined over the rectangular control grid $\mathbf{V_{k,j}}$, k=0..N, j=0..M:

$$S = \sum_{i=j=0}^{N,M} S_{k,j} \qquad (7)$$

3. The topology of spline curve

We can say, that a spline curve "copies" the shape of the *control polygon* $\mathbf{V_0,V_1,...V_N}$. From the topological aspect it is interesting, how the control polygon is designed by multiple control points or by the cycle of points. We describe this in detail:
Let A, B, C, D, E be different control points. Then:

1. the segment S(A,A,A,A) is the point A,
2. the segment S(A,A,A,B) is a line segment and A is its boundary point,
3. the segment S(A,A,B,B) is a line segment,
4. the segment S(A,A,B,C) is formed of a curve,
5. if we have a cycle of points (first three and last three points are same) e.g. ABCDABC, then according to (3) the spline is a smooth close curve.

4. Modelling of some basic topological surfaces

In previous works several approaches to arbitrary topology surfaces have been suggested [CC78] [Loo94][GH95]. These approaches enable practical modelling of the surfaces with a various topology, but they can't design some topological surfaces like the Moebius strip or the Klein bottle. We suggest a very simple approach to model some basic topological surfaces using binding sides of the control grid of the B-spline surface. This approach enables modelling of the Moebius strip and the Klein bottle, but it isn't suitable for modelling of complex surfaces.

The main idea comes out from a construction of some topological surfaces. When we have the square **A**:{ $(x,y)\in[0,1]x[0,1]$} we can from this square construct some topological surfaces by using formulas (8),..(12). (Fig 1) This construction is analogous to the production of cake from a rectangular piece of a dough by gluing two sides of the rectangle.

Topological construction of some surfaces from the square A:

square **A**:{ $(x,y)\in[0,1]x[0,1]$}

1. the equivalency (1,y) = (0,y) result is a cylinder, (8)
2. the equivalency (1,y) = (0,1-y) result is the Moebius strip, (9)
3. the equivalency (1,y) = (0,y) and (x,1) = (x,0) result is a torus, (10)

4. the equivalency (1,y) = (0,y) and (x,1) = (1-x,0) result is the Klein bottle, (11)
5. the equivalency (0,y) = (x,0) and (1,y) = (x,1) result is a sphere. (12)

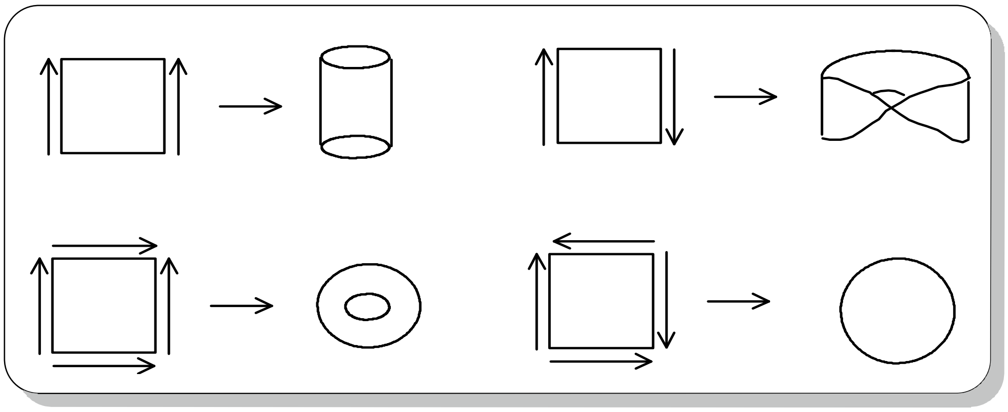

Fig.1 Topological construction of some surfaces from the square.

Back to the spline modelling. In B-spline curves we mentioned, that they copy the shape of the control polygon. For B-spline surfaces according to (7) the control grid must be rectangular. Therefore modelling surfaces, which topology is different from a rectangular is not simple. To overcome this restriction we use multiple control points or cycles of points. This method will inherit many properties of B-spline surfaces: the local control, the compact representation, and the continuity of second degree. The surface will be created by u-curves and v-curves. It will be not difficult to map the texture on the surface. Generally, if we cannot model the topological difficult surface, then we must divide this surface to any topological more simple surfaces, and then together smoothly.

In our modelling method we try to use an approach analogous to the topological construction of some surfaces from the square.

5. Cylinder

We design cylinder like to topological construction of a cylinder. If we have replaced the square A [0,1]x[0,1] with the rectangular grid of control points, and the equivalency (8) with the condition (13), then we obtain a smooth cylinder surface.

Condition for cylinder surface:

The first three and the last three columns (rows) are the same. (13)

Construction of cylinder surface:

Add the first three columns (rows) to end of control grid. (14)

U-curves will be cycles, and v-curves will be line segments (in the topological aspect) (Ex.1, Fig. 2)

0	1	2	3	0	1	2
0	1	2	3	0	1	2
4	5	6	7	4	5	7
4	5	6	7	4	5	7

Ex.1 The Control grid of a cylinder surface (the numbers are labels of control points).

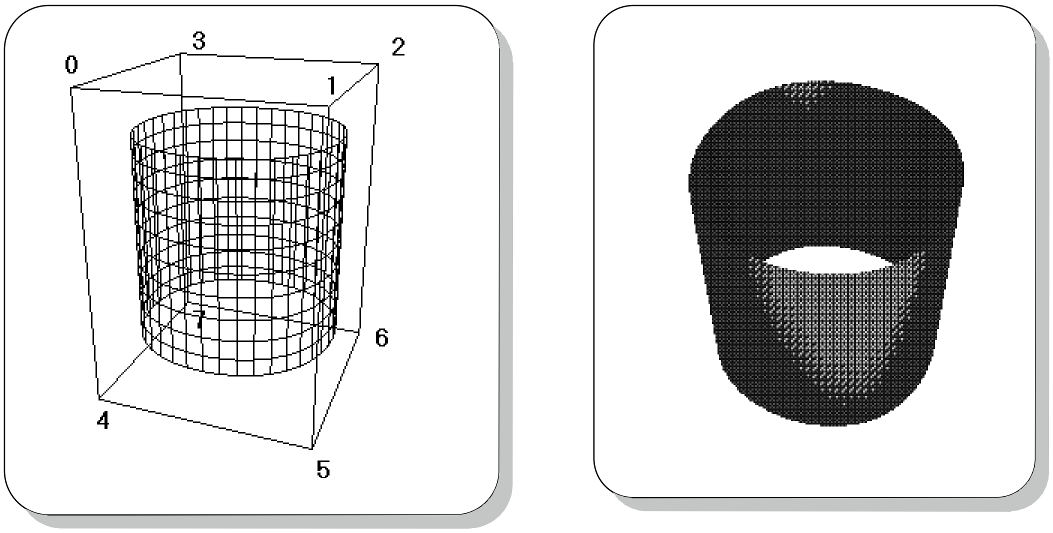

Fig.2, Cylinder (8 control points)

6. Torus

If we use (14) for all opposite sides of control grid then we will obtain a torus. (Ex. 2, Fig.3)

1	2	3	4	1	2	3
5	6	7	8	5	6	7
9	10	11	12	9	10	11
13	14	15	16	13	14	15
1	2	3	4	1	2	3
5	6	7	8	5	6	7
9	10	11	12	9	10	11

Ex. 2, Control grid of torus

Condition for a torus:

First three and last three rows are same, and the first three and last three columns are the same too. (15)

Construction of a torus:

Add first three rows to the end of the control grid, then add the first three columns to the end of control grid. (16)

This construction is formally commutative. (If we add the first rows or columns the resulting control grid is the same.) U-curves and v-curves are topologically circles.

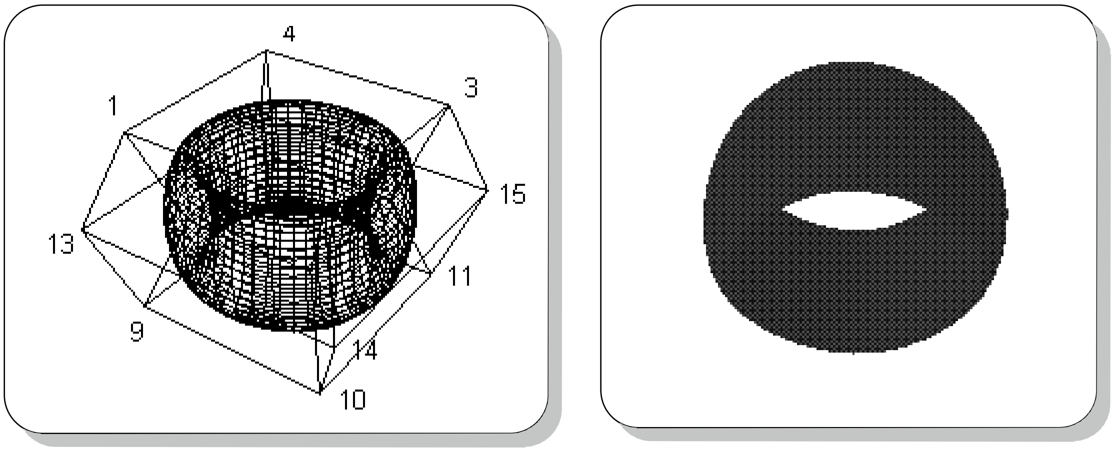

Fig. 3, Torus (16 control points)

7. Moebius strip

The modelling of the Moebius strip is similar to gluing of a paper strip. The first three and the last three rows (columns) are the same in inverse order. This condition is analogous to (9). The Moebius strip is not an orientable surface.

Condition for the Moebius strip:

The first three and the last three rows are the same in reversed order. (17)

Construction of the Moebius strip:

Add the first three rows (columns) to the opposite side of the control grid in reversed order. (18)

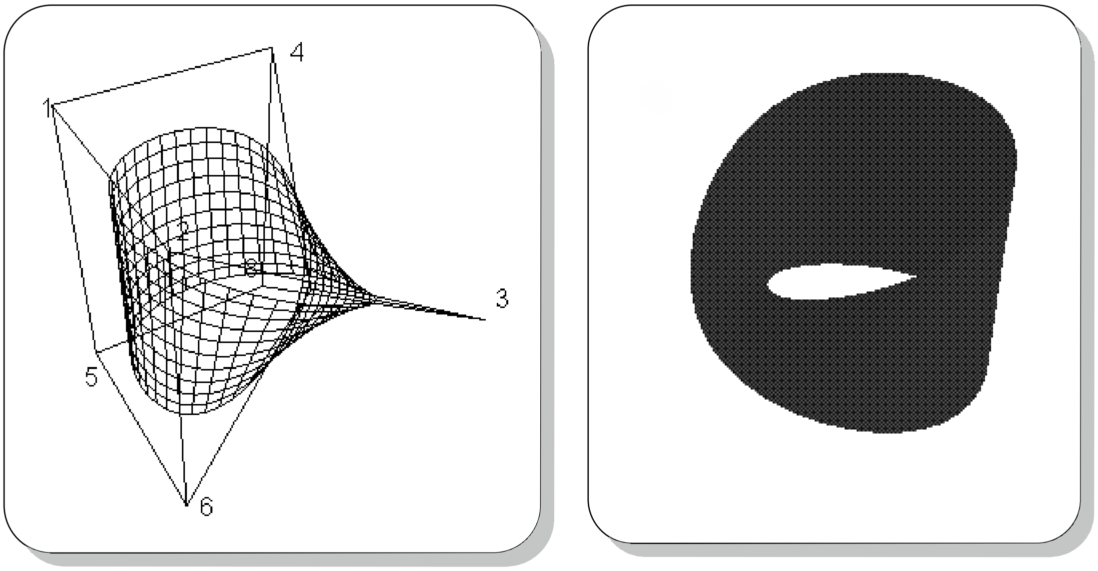

Fig.4 Moebius strip (8 control points)

1	2	3	4	5	6	7
1	2	3	4	5	6	7
5	6	7	8	1	2	3
5	6	7	8	1	2	3

Ex.3, Control grid of Moebius strip

8. Klein bottle

If we have used the construction (14) and for remaining two opposite sides the construction (18) we obtain the control grid of the Klein bottle. The Klein bottle is a nonorientable surface, and it is impossible to display this surface without a self-intersection. (Ex.4, Fig. 5,6)

Unlike the (16) this approach is not formally commutative. The resulting control grid depends, whether we have used first the construction (14) or (18).

Construction of Klein bottle:

Add the first three columns to the end of the control grid, then add in reversed order the first three rows to the end of the control grid. (19)

or

Add in reversed order the first three rows to the end of the control grid, then add the first three columns to the end of the control grid. (20)

Although the resulting control grids are different, the designed shapes over these control grids are the same.

1	2	3	4	1	2	3
5	6	7	8	5	6	7
9	10	11	12	9	10	11
13	14	15	16	13	14	15
3	2	1	4	3	2	1
7	6	5	8	7	6	5
11	10	9	12	11	10	9

1	2	3	4	1	2	3
5	6	7	8	5	6	7
9	10	11	12	9	10	11
13	14	15	16	13	14	15
4	3	2	1	4	3	2
8	7	6	5	8	7	6
12	11	10	9	12	11	10

Ex.4, Control grids of the Klein bottle.

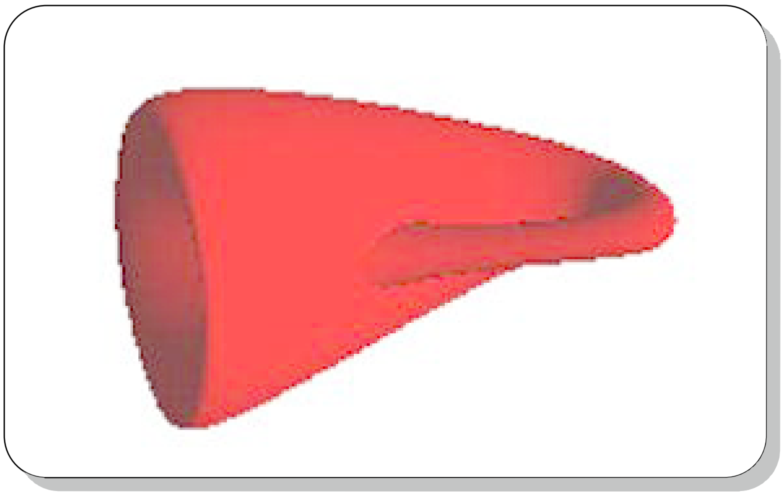

Fig.5, Klein bottle (16 control points)

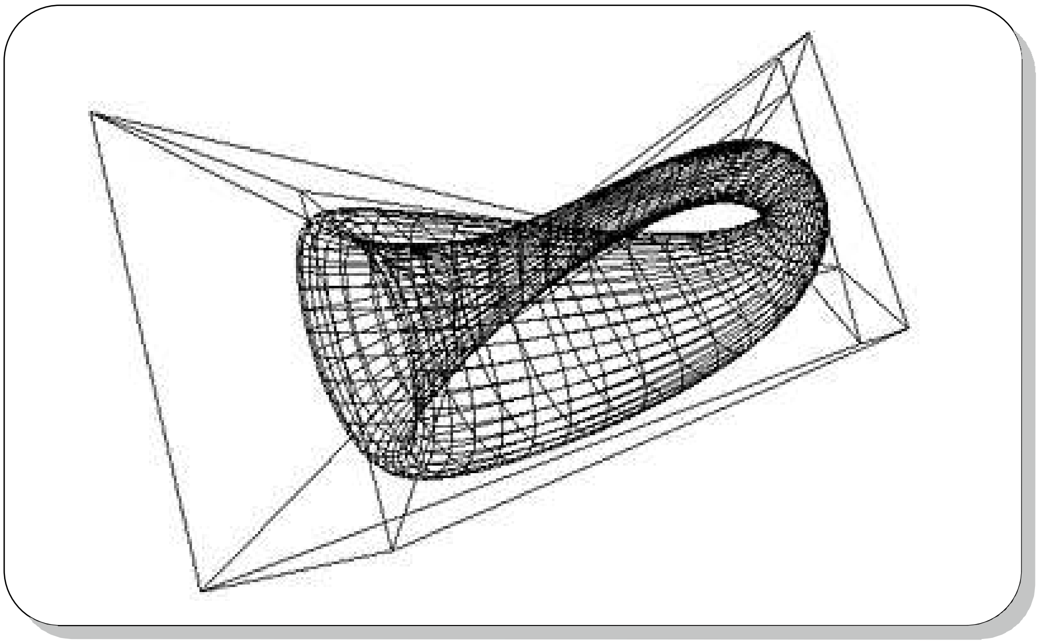

Fig.6, Klein bottle - wire model (16 control points)

9. Sphere

Till now we haven't any problems with the analogy of the modelling B-spline surfaces and the topological construction some surfaces. We can topologically construct a sphere by pasting together the adjacent sides are (12). But the condition that the first three rows and the first three columns are identical leads to the distortion of continuity and parameterization in the corner point of the control grid. In the Fig 7. we have an example of a this way surface designed.

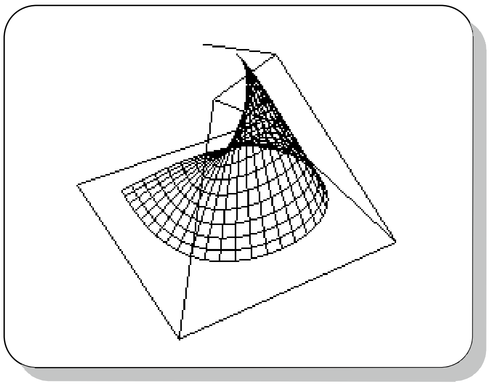

Fig. 7

To design a simple sphere surface we can use construction of the cylinder surface (14), so that all v-curves begin and end in the same points - poles. This sphere modelling is identical to a world globe. The v-curve are meridians and the u-curve are parallels. To retain the smoothness in the poles, we need the restriction, that poles and surrounding control points are in the same plane. (Ex. 5, Fig. 8)

N	N	N	N	N	N	N
N	N	N	N	N	N	N
N	N	N	N	N	N	N
1	2	3	4	1	2	3
5	6	7	8	5	6	7
S	S	S	S	S	S	S
S	S	S	S	S	S	S
S	S	S	S	S	S	S

Ex. 5, Control grid of sphere

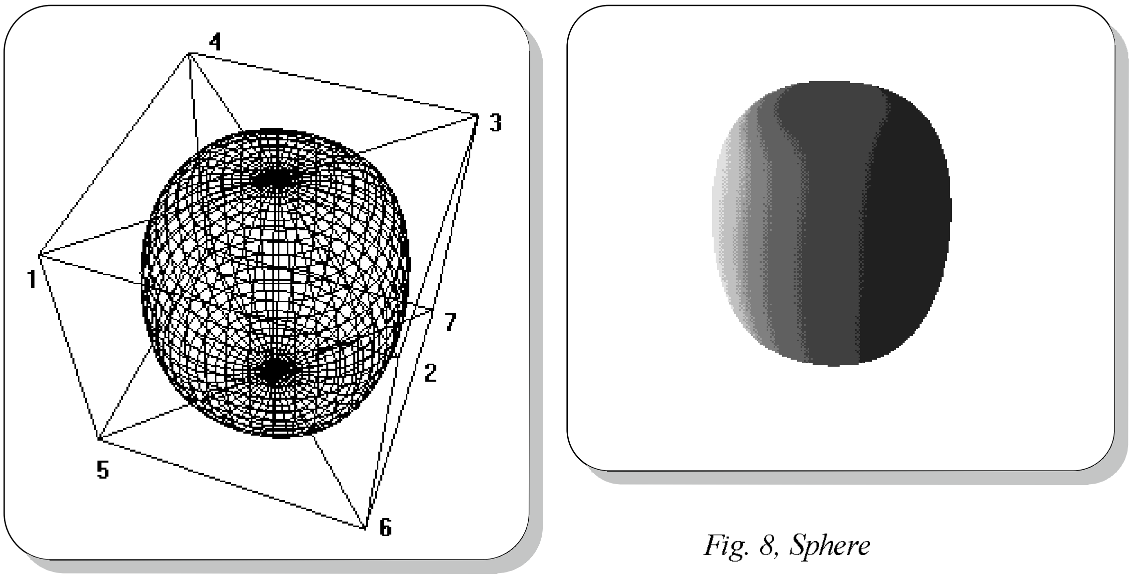

Fig. 8, Sphere

10. Remarks

Examples of control grids 1., 2., 3., 4. and 5. describe topological properties of designed surfaces, but their shapes depend mainly from positions of control points. The shape of control polyhedron must correspond to the expected shape of the B-spline surface.

In our examples and figures we tried to use minimal number of control points to model some basic types of topological surfaces. The shift of one control point will smoothly change the shape of the whole surface. We can use this property of control points to create a simple 3D animation.

Each point of the designed spline surface has u- and v-coordinates, therefore it is easy to trace or map texture on this surface and there are other benefits too.

Designed surfaces are evaluated in Bezier vertices of the Bezier surface. We made the visualisation of shown surfaces as visualisation of Bezier surfaces in OpenGL on PC (486, Pentium).

The Moebius strip and the Klein bottle are not orientable surfaces, therefore for a light model we must use surface normals independent of the orientation.

In the future work we can extend this method to other spline types. Also we can loose the restriction, that the control grid must be rectangular.

Literature

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