3D structured meshes for the FEM

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Abstract. Special type of mesh – a structured (regular) mesh generated from the rectangular index grid – is discussed in context of the finite element modelling.

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1 Introduction

This paper deals with some problems arising in the course of developing a preprocessor of a finite element code based on structured meshes. More precisely, it concerns the effect of the way of decomposition of hexahedral bricks into tetrahedra on the accuracy of the solution.

Structural mesh we understand a mesh which can be obtained as a result of deformation of the regular rectangular mesh. The use of structural meshes is advantegeous for various reasons, mainly:

- illustrativity of the mesh, especially in 3D: we can enumerate nodes using 3 indices; nodes can be then naturally grouped to sets – *index lines* or *index planes* with two or one constant indices respectively
- simplicity of algorithms for mesh generation,
- regular data structures, which result in efficient numerical procedures for solving the problems.

Despite the above advantages, a structured mesh is a bit less flexible than an unstructured one. Nevertheless, there are many classes of problems where this disadvantage is not important. As an example, we can mention problems arising from modelling the effect of mining in coal seams. Let us note, however, that the structured grid may be more flexible than can be expected at the first glance, see [2] and Fig.1. Moreover, the lack of flexibility in modelling problems, which require a local refinement of the mesh, can be removed by the use of composite structured meshes. Numerical solutions of problems on composite meshes can be efficiently found by various variants of the FAC (Fast Adaptive Composite grid) technique by S. McCormick, see, e.g., [1]. This procedure also enables an adaptive refinement.

2 Mesh generator

Our structured 3D mesh generator proceeds in two steps:

- the deformation of the regular rectangular mesh, conserving the convexity of hexahedral bricks,
- the decomposition of hexahedral bricks into tetrahedra.

A 2D analogy of this procedure can be seen in Fig.1. Note that an additional flexibility of the mesh is obtained by permitting some tetrahedra of the final decomposition to be void.

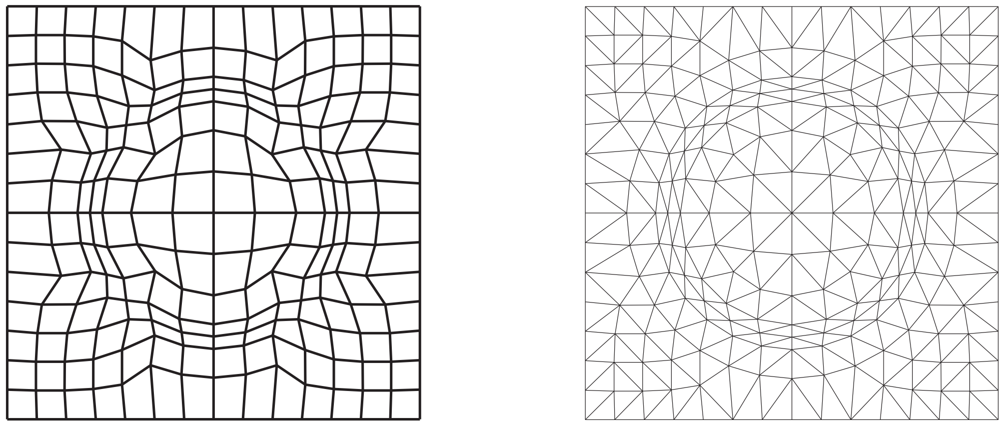

Fig. 1. Example of a structured mesh: a. – the deformation of the regular rectangular mesh; b. – the decomposition into triangles using diagonals

A hexahedral brick can be divided into five or six tetrahedra. The latter subdivision is chosen with respect to gradient recovery procedures which can be used in this case, see [4].

3 Decompositions into tetrahedra

An important role is played by *conform decompositions*, i.e. decompositions where the intersection of the neighbour elements is a node or a whole edge or a whole face.

The most important properties concerning the decompositions of a convex hexahedral brick into six tetrahedra are [3]:

- face diagonals cannot be chosen arbitrarily: in the situation in Fig. 2a there exist only nonconform decompositions; in the situation in Fig. 2b there exists no decomposition.
- in general, the face diagonals are not sufficient for determination of the conform decomposition — the information about the body diagonal is necessary; only for the decompositions from Fig. 3c, 3d the face diagonals determine the decomposition;

- there are 72 conform and 84 nonconform decompositions, see Fig. 3;

a. b.

Fig. 2. Face diagonals, for which there exists no conform decomposition.

An algorithm which minimizes the nonconformity, which is sometimes necessary due to inner boundaries given by material nonhomogeneity, is based on the properties mentioned above and proceeds in the following steps:

1. indicating the face and body diagonals of hexahedral bricks determined by the geometry of the domain of the problem,
2. choosing the rest of face diagonals avoiding the situation in Fig. 2a, 2b; (if it is impossible, the brick is decomposed nonconformly)
3. adding the rest of body diagonals,
4. identifying the decomposition by face and body diagonals.

As usual, only a few diagonals are used for the geometry determination (step 1), so we have many possibilities how to fill the rest of face diagonals (step 2) and body diagonals (step 3). Below we describe which decomposition we prefer.

4 Brick decompositions and FE approximation

We attempt to clear up the effect of various ways of decomposing the bricks into six tetrahedra on the accuracy of the finite element solution. To this end we will consider a linear elasticity problem characterized by:

- the unit cube domain $\Omega = (0,1)^3$,
- isotropic material with Lamé moduli $\lambda = 1$ and $\mu = 1.5$,
- homogeneous Dirichlet boundary conditions along the whole boundary $\partial\Omega$,
- a volume force giving the exact solution of the form
  $u_1(x_1,x_2,x_3) = u_2(x_1,x_2,x_3) = x_1x_2x_3(1-x_1)(1-x_2)(1-x_3)$,
  $u_3(x_1,x_2,x_3) = \sin\pi x_1 \sin\pi x_2 \sin\pi x_3$.

Next types of decompositions are used:

Type 1 – a conform decomposition; all bricks are decomposed according to Fig. 3d. In this case the superconvergence is achieved, see [4].

Type 2 – similar to type 1, except of one brick, which is decomposed nonconformly, according to Fig. 3f. Note that nonconformity arises also on the faces of the brick.

Type 3 – a conform decomposition, different from type 1; all bricks are decomposed according to Fig. 3e.

Type 4 – similar to type 3, except of one brick, which is decomposed nonconformly, according to Fig. 3f. Note, that nonconformity is only inside the brick.

Type 5 – all bricks are decomposed nonconformly according to Fig. 3f. Nonconformity exists only inside the bricks.

Fig. 3. a. – e. Conform decompositions of a cube to six tetrahedra. All conform decompositions are obtained from these by rotation. f. – Example of a nonconform decomposition.

We use l_2 and max norms for the comparison of differences between the exact and computed displacements u. Nit is the number of iterations of the conjugate gradient method exploited for the solution of the FE system and nn denotes the number of nodes in the mesh.

*	1	2	3	4	5	nn
Nit	9	10	11	12	12	8x8x8
$l_2(u)$	.323709	.324760	.414579	.415056	.487083	
$max(u)$	.0445725	.0504606	.0569136	.0566414	.0666274	
Nit	19	20	23	24	23	15x15x15
$l_2(u)$	.23986	.239805	.312405	.312404	.370027	
$max(u)$	.0132336	.0152178	.0166554	.0177262	.019420	
Nit	42	43	51	52	51	29x29x29
$l_2(u)$	.175620	.174779	.228524	.228416	.280246	
$max(u)$	.00382435	.00432533	.00475687	.00509328	.00552094	

From the above table we can observe that the way of decomposing the hexahedral bricks into tetrahedra has surprisingly strong influence on the accuracy of the solution and on properties of the corresponding stiffness matrix. We can also observe a relatively large difference between the two different conform subdivisions (type 1 and type 3) which is larger then the effect of local nonconformity (difference between type 1 and 2, type 3 and 4). Even for a fully nonconform subdivision we can observe the convergence of the finite element approximation. We can also see the influence of the way of decomposing the hexahedral brick on the spectral properties of the stiffness matrix which is reflected by the number of iterations of the conjugate gradient method. From the above observations, the importance of division of hexahedral bricks into tetrahedra becomes apparent.

5 Conclusions

The way of decomposing the hexahedral bricks into tetrahedra has a strong influence on the accuracy of the solution and on the properties of the corresponding stiffness matrix. Conform decompositions play an important role here.

The described ideas of using structured meshes with conform decompositions for tetrahedral elements have been implemented in GEM32 code for the FE analysis of 3D problems of elasticity and plasticity.

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