Constructive Geometry of Conics Principal Conjugations

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Abstract

Curves of the second order or conics are the most important geometric images on which many provisions of projective geometry are built. Despite the development of this scientific discipline, there are still many issues that require closer study and research. The constructive geometric method makes it possible to discover new previously unknown properties that appear in conics both in independent images and in complexes of geometric images that can be represented by these curves. This approach allows us to give a new projective interpretation to many problems to which it has not been applied before. An example of such a problem, in particular, is the problem of constructing a line equidistant with respect to two circles (or spheres of higher-dimensional spaces). The constructive geometric method allows us to find the most general ways to solve geometric problems, and their extension to the region of imaginary geometric values makes it possible to get rid of annoying exceptions. The development of the theory of curves of the second order contributes to the improvement of information technologies based on automation tools for solving constructive geometric problems.

Keywords

Constructive Geometric modeling, projective geometry, geometric experiment, second-order Curve diameters, focal points, conic conjugations, simplex

1. Introduction

Imaginary images often accompany the solution of geometric problems if these problems need to be presented in a general form that avoids exceptions. In mathematics, imaginary objects have been used successfully and for a long time. The theory of imaginary numbers is based on the introduction of the concept of an imaginary unit – the result of extracting the square root of a number minus 1. The algebra of imaginary numbers practically does not differ from the algebra of real numbers. The only difference is that real quantities are represented directly by real numbers, while imaginary (complex) quantities are represented by two real numbers. And to this are added the rules for manipulating these numbers, which allow us to obtain the result necessary and justified from the point of view of algebra.

A similar principle of working with imaginary geometric images should be used in geometry, however, constructive algorithms that solve applied and theoretical problems have not yet been fully developed. This explains the interest of researchers in the problems of specifying second-order curves with the help of imaginary images [6–11, 15, 17–25]. The need for such algorithms is explained by the need to create new information technologies, improve systems of automation tools for geometric constructions [16] and expand the capabilities of CAD software [2]. In this regard, the task of identifying new patterns that make it possible to implement new functionality of such systems is relevant.

It is impossible to depict the imaginary image directly in the drawing. However, by acting indirectly with the help of real objects that allow graphical representation, imaginary geometric images can be made available for use in the usual practice of geometric modeling. Moreover, imaginaries in practical solutions in the overwhelming majority of cases arise from the interaction of real images. Therefore, the most natural way to represent imaginary images is to replace them with real images with the addition of operations that generate a solution in the imaginary representation. An analysis of the constructive connections that arise in the process of experimenting with geometric algorithms allows us to give new

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interpretations of long-established geometric concepts that cannot be traced in the analytical representation of geometric problems [1, 3].

This article is devoted to a number of issues related to an attempt to justify the geometric nature of the diameters of conics, the representation of the axes of these curves, the study of complexes of principal conjugations of the conic triad system, and the proposal of a new projective interpretation [14, 27] of the problem of constructing equidistant with respect to two circles [12, 13]. The algorithms presented in the article are the basis for setting and solving similar problems on projection models of spaces of higher dimensions [4, 5, 26] and are the mathematical basis for improving the functional composition of the Simplex system developed by the author.

2. The constructive geometric nature of images associated with conic sections

2.1. Construction of principal conjugations of a conic

Let a conic a be given. It is required to construct its principal conic conjugations b and c.

Let's build axes u and v of conics a. Let's find the points $A,C = a \times u$, $B,D = a \times v$.

Let's construct an arbitrary circle a' in the plane and draw mutually perpendicular axes u' and v' through its center Ca'

Let's find points $A',C' = a' \times u'$, $B',D' = a' \times v'$.

Let us define a collinear transformation $\xi \mid \begin{matrix} A;B;C;D \\ A';B';C';D' \end{matrix}$ on the plane. This transformation transforms the conic a into a circle $a' = \xi(a)$.

Let's construct the tangents to the circle a at the points A', B', C' and D': $p' = \Pi_{a'}(A')$, $q' = \Pi_{a'}(B')$, $r' = \Pi_{a'}(C')$ and $s' = \Pi_{a'}(D')$.

Let us draw bisector lines m' and n' with respect to the axes u' and v' of conics a' to determine the directions to the infinitely distant points M'^{∞} and N'^{∞}.

Let's build conics $b' = A' \circ C' \circ M'^{\infty}$; $A' \sim p'$; $C' \sim r'$ and $c' = B' \circ D' \circ N'^{\infty}$; $B' \sim q'$; $D' \sim s'$.

We are looking for the desired conjugations as $b = \xi^{-1}(b')$ and $c = \xi^{-1}(c')$ (figure 1).

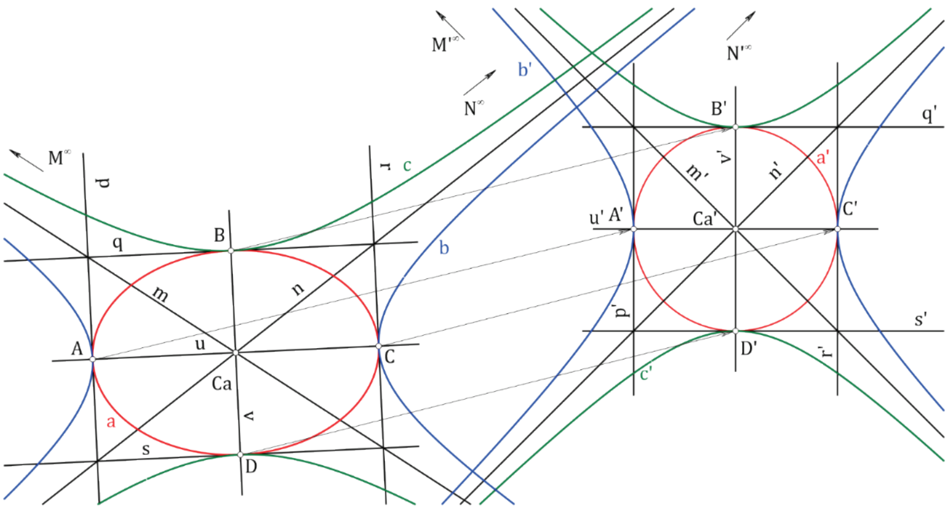

Figure 1: Reducing conjugations to canonical form

It can be seen from the presented scheme that conics m' and n' are common asymptotes of conics b' and c', therefore, lines $m = \xi^{-1}(m')$ and $n = \xi^{-1}(n')$ are common asymptotes of conics b and c. At the same time, the straight lines $p = \xi^{-1}(p')$, $q = \xi^{-1}(q')$, $r = \xi^{-1}(r')$, $s = \xi^{-1}(s')$ are common asymptotes of the conics b and c, and the points $M^{\infty}=\xi^{-1}\left(M'^{\infty}\right)$ and $N^{\infty}=\xi^{-1}\left(N'^{\infty}\right)$. Note that in the above diagram $A,B,C,D\sim a$; $A,C,M^{\infty},N^{\infty}\sim b$; $B,D,M^{\infty},N^{\infty}\sim b$.

2.2. Asymptotes of conics

According to the generally accepted definition, the asymptote of a curved line with an infinite branch is understood to be a line that does not coincide, does not touch it, and has the property that the distance from the point of the curve to this line tends to zero when the point is removed along the branch to infinity. In projective geometry, in which there is no concept of distance, the asymptote of a conic (hyperbola) is understood as a straight line tangent to this curve at the point of intersection with a straight line at infinity. In other words, the asymptote is the polar for such a point with respect to the given curve. Having two points of intersection with a straight line at infinity, a hyperbola generates two asymptotes that intersect at the center point of this hyperbola. Since the ellipse has no obvious points of intersection with the infinite line, it is believed that it has no asymptotes. This assertion, however, leads to a contradiction, which arises from another assertion, which says that second-order curves in projective geometry are indistinguishable. Therefore, properties based on their incidence with other objects should not differ either.

Let there be some ellipse a and some hyperbola b. Denote by u and v the principal diameters of the ellipse and find the points $A_1,A_3=a\times u$ and $A_2,A_4=a\times v$ of their intersections with the conic a. Let also $B_1,B_3=b\times q$ be the points of intersection of the principal diameter q of the conic b, and the points B_2 and B_4 – be the points of intersection of the hyperbola b with the straight line at infinity. Having defined the collinear transformation $\xi\,|\,\frac{A_1;A_2;A_3;A_4}{B_2;B_1;B_3;B_4}$, we get the possibility of transformation $a'=\xi\left(a\right)$, while it is found that $a'\equiv b$, $u'=\xi\left(u\right)$; $q\equiv u'$, and $v'=\xi\left(v\right)$ becomes a real straight line at infinity. Thus, we can assume that the lines $a_1\sim A_1\sim a$; $a_2\sim A_2\sim a$; $a_3\sim A_3\sim a$; $a_4\sim A_4\sim a$ and $a_1'=\xi\left(a_1\right)$; $a_2'=\xi\left(a_2\right)$; $a_3'=\xi\left(a_3\right)$; $a_4'=\xi\left(a_4\right)$ are in a constructive-geometric relationship, performing the general functions of tangents at the points of intersection of conics with principal diameters, and can be conventionally called asymptotes of conics in their projective manifestation (figure 2).

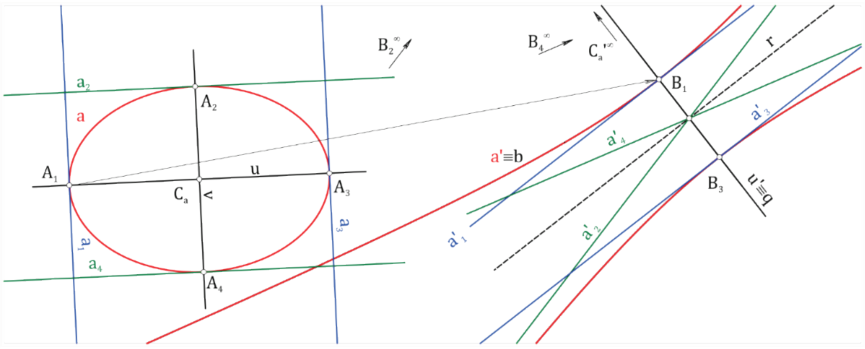

Figure 2: Extended representation of the asymptotes of conics

We investigate the question of the possibility of using the imaginary diameter to solve the problem of collinear transformation of a conic into another conic (figure 3).

Let two conics be given in the plane in the form of some ellipse a and some hyperbola b. Let's define the principal diameters of the hyperbola – real v' and "imaginary" u'. Denote by $B_1,B_3=b\times v'$ and $\overline{B}_2,\overline{B}_4=b\times o_3$. Let us establish a collinear transformation $\chi\,|\,\frac{A_1;A_2;A_3;A_4}{B_1;\overline{B}_2;B_3;\overline{B}_4}$ on the plane. This collinear transformation takes the real conic a to $a'=\chi\left(a\right)$; $a'\equiv b$, and the principal diameters u and

v into the principal diameters $u' = \chi(u)$, $v' = \chi(v)$ of the conics $a' \equiv b$. Let's construct tangents to the conic a' at the points B_2 and B_4: $a_2' = \chi(a_2)$, $a_4' = \chi(a_4)$. As can be seen from the drawing, both tangents will intersect at an infinitely distant point $T^\infty = a_2' \times a_4'$ in the direction of the real diameter v'. It should be noted that the direction to the point does not change if the collineation $\chi \left| \begin{matrix} A_1;A_2;A_3;A_4 \\ B_1;\bar{B}_2;B_3;\bar{B}_4 \end{matrix} \right.$ is redefined to $\chi' \left| \begin{matrix} A_1;A_2;A_3;A_4 \\ B_2;B_3;B_4;B_1 \end{matrix} \right.$ and the previously presented constructions are performed taking into account the new collineation (however, the mutual correspondence of the diameters will change).

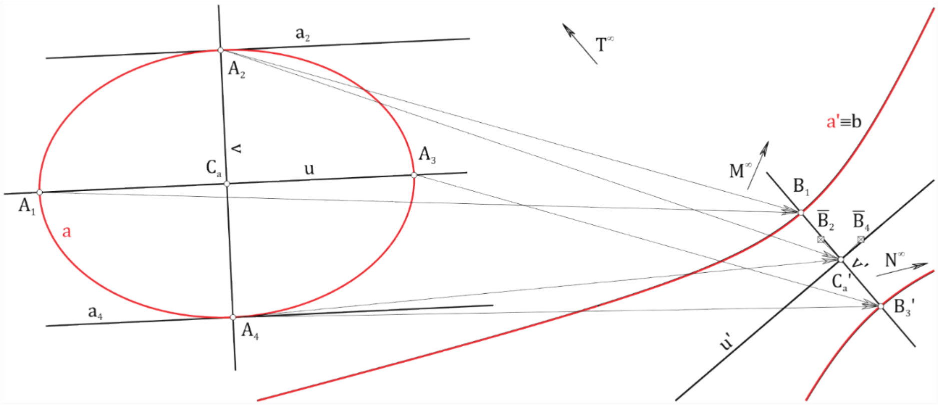

Figure 3: Center of a conic in a collinear transformation

It can be seen from the above constructions that a real proper point Ca' and an infinitely distant improper point T^∞ have similar geometric manifestations, and when collineations are established that transform conics into conics, they can transform into each other. On the other hand, points at infinity, located along the direction of the principal diameters u and v will be transformed into points C_a' and at infinity, located along the direction of the "imaginary" diameter of the hyperbola (for the example under consideration), changing places depending on the type of collineation χ or χ'. All this indicates an inextricable constructive connection between the points under consideration and their belonging to a single class of centers of curves of the second order with the only difference – proper or non-proper.

The suggestions made are confirmed by the following constructive scheme. figure 4 shows a bundle of conics passing through the centers A, B, C, D and the centers of these conics are constructed. Through the centers of the conics, a second-order curve is drawn, represented by a hyperbola, which bears both the proper centers of the conics of the bundle and two improper points.

2.3. Justification of the simplified construction of the ellipse's foci

Let an ellipse e be given on the plane (figure 5). We construct the principal diameters u and v. We choose an arbitrary point $P \sim e$ on the ellipse and draw a straight line $p = \Pi_e(P)$ through it, tangent to the ellipse e. Restore the perpendicular $p' : p' \perp p, p' \sim P$ and find the points $A = p \times u$, $A' = p' \times u$, $B = q \times u$, $B' = q' \times u$. We define an involution $\zeta \left| \begin{matrix} u;A;B \\ u;A';B' \end{matrix} \right.$ on the line u and find its fixed points F_1 and F_2, which are focal points of the ellipse . Let us pay attention to the fact that the circles a and b intersect each other at imaginary complex conjugate points $\bar{F}_3, \bar{F}_4 = a \times b$, which by their geometric nature are the imaginary foci of the ellipse e. The points $\bar{F}_3$ and $\bar{F}_4$ are the imaginary centers of the bundle of circles $(a, b, ...)$, while the points F_1 and F_2 are the centers of another bundle orthogonal to the bundle $(a,b,...)$. The problem is to find an indirect way of constructing the points F_1 and F_2 that does not require projective functions to determine them.

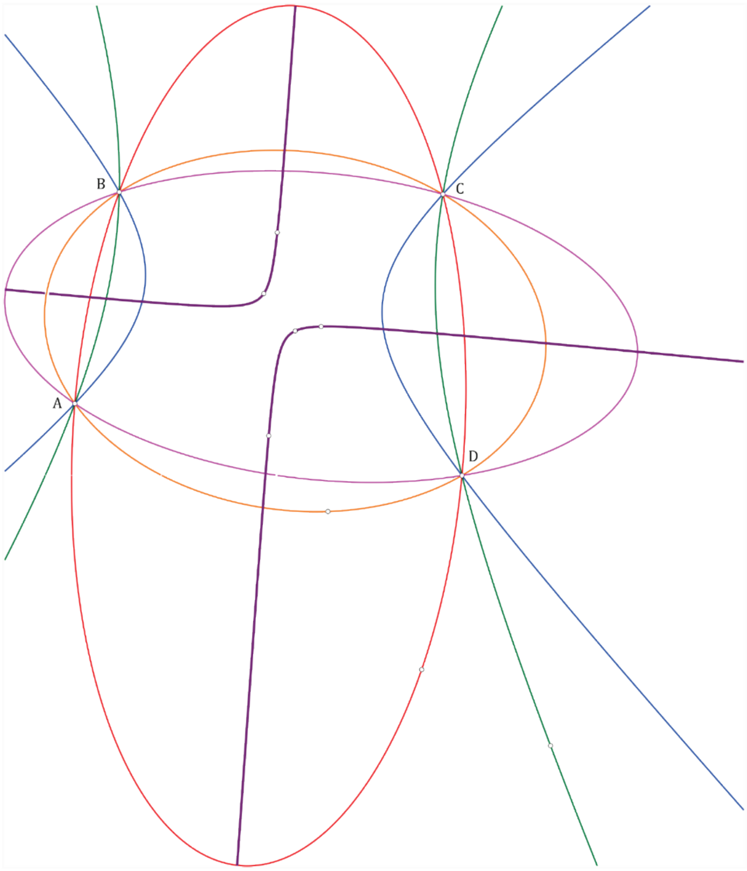

Figure 4: The conic-carrier of the centers of a bundle of conics

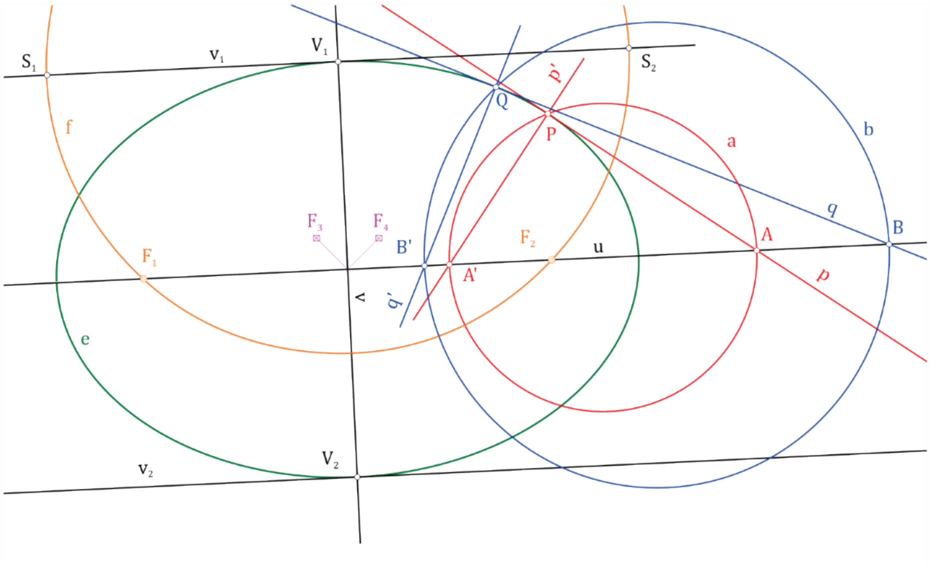

Figure 5: Projective scheme for constructing of a conic's foci

Let's draw a straight line $v_1 : v_1 \perp v, v_1 \sim V_1$. To select a single circle from the bundle with centers F_1 and F_2 we construct a circle $f : f \perp b, f \perp a, f \perp v_1$. It is easy to see that the points $S_1, S_2 = f \times v_1$ lie at the intersections of the asymptotes of the ellipse, which can be easily constructed without resorting to projective functions. Thus, to construct the real foci of the ellipse, it is necessary and sufficient to construct the diametral points S_1 and S_2 and, drawing a circle f through them, at the intersection with a straight line u, mark the focal points F_1 and F_2. This method of foci constructing is well known and widely used in practice.

Similar reasoning could be carried out with regard to the involution defined on the line v, however, in essence, the foci $\overline{F}_3$ and $\overline{F}_4$ are already known and they can be obtained by indicating the points of intersection of any two circles from the bundle $(a,b,...)$ or any circle from this bundle with a straight line v. However, to perform this operation, such circles need to be built. You can specify another way to solve the problem. It is easy to see that null-circles with centers at known points F_1 and F_2 also belong to the bundle $(a,b,...)$. Denoting these circles as f_1 and f_2 we find the points $\overline{F}_3$ and $\overline{F}_4$ as points of intersection of the zero-circles f_1 and f_2: $F_3, F_4 = f_1 \times f_2$. A similar reasoning can be carried out with respect to the orthogonal bundle $(f,...)$. Drawing two imaginary zero-circles $\overline{f}_3$ and $\overline{f}_4$ with centers at imaginary complex conjugate points $\overline{F}_3$ and $\overline{F}_4$ and, finding their intersection points, we obtain real points F_1 and F_2: $F_1, F_2 = f_3 \times f_4$. Since the focal points are two centers of a bundle of circles that also intersect at imaginary cyclic points, these points, which have a constructive nature common with real points, can also be referred to as focal points of a second-order curve.

2.4. Some properties of conjugations of conics

2.4.1. The real foci of conjugate hyperbolas are located on the same circle

Let a complex of conjugate conics $a-b-c$ be given on the plane. A conic a is an ellipse, conics b and c are hyperbolas. We study the focal properties of the objects of this system.

Let us draw asymptotes k and l of conic b. In accordance with 2.3, in order to construct real foci of a conic b, it is necessary to draw a circle f through points $Q = a_1 \times a_2$ and $S = a_3 \times a_4$ or points $P = a_1 \times a_4$ and $R = a_2 \times a_3$ taken as diametrical.

Let us draw asymptotes k and l of conic c. Due to the fact that the asymptotes k and l of conics c and b coincide, we come to a similar construction: in accordance with 2.3, to construct the real foci of a conic c, it is necessary to draw a circle f through points $Q = a_1 \times a_2$ and $S = a_3 \times a_4$ or points $P = a_1 \times a_4$ and $R = a_2 \times a_3$ taken as diametrical. Then at the intersection with the straight line v, which is the real diameter of the conic c, we obtain real focal points Fc_1 and Fc_2 of conic c.

2.4.2. The lines connecting the real foci of adjacent conjugate hyperbolas are tangent to the conjugate ellipse

Let $db_1 = \Pi_b(Fb_1)$, $dc_1 = \Pi_c(Fc_1)$, $db_2 = \Pi_b(Fb_2)$ and $dc_2 = \Pi_c(Fc_2)$. Let also $e = \Pi_a(f)$. By virtue of the incidence principle, the straight lines db_1, dc_1, db_2, dc_2 are asymptotes of the conic e. This means that the tangents dropped to the conic a from the foci Fb_1, Fc_1, Fb_2, Fc_2 have common points with the intersection points of the asymptotes of the conic e. Therefore, the lines p, q, r, s form a square with vertices Fb_1, Fc_1, Fb_2, Fc_2, incident with the circle f.

2.4.3. The image of a conic in polarity with respect to the conjugate conic coincides with its preimage

Any two conics from the system of three conjugate conics have two adjacent and two non-adjacent asymptotes. The transformation of adjacent asymptotes and common tangent points in polarity with respect to any of the adjacent conics leads to the formation of the same line-tangent pair. A polar transformation of a non-adjacent asymptote and a tangency point of one conic with respect to another conic of a pair leads to a pairwise non-adjacent asymptote and a tangency point of the first conic. These conditions are more than enough to prove the fact that a conic is transformed into itself.

2.4.4. The system of three conjugate conics is preserved under any collinear transformation of the conics of one pair

The proof of this fact is exceptionally simple and is based on mutual mapping for a given collineation of the asymptotes of the system of conjugate conics into each other (figure 6).

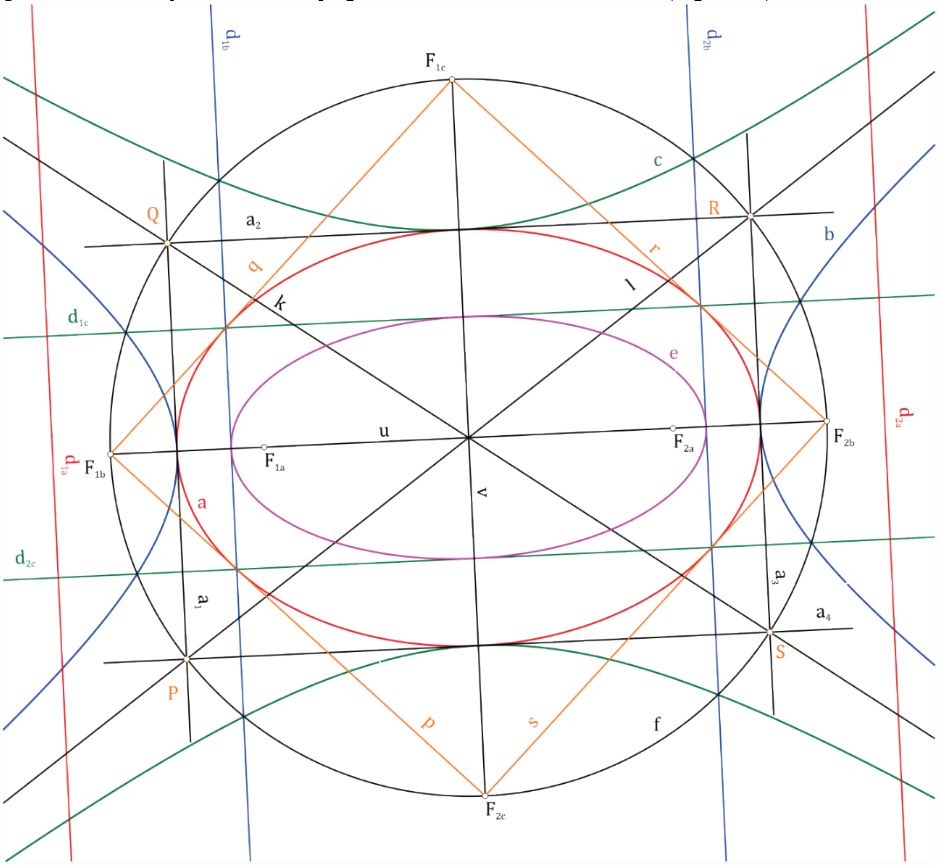

Figure 6: Conics principal conjugations

2.4.5. The conic is visible from any point on the focal circle at a right angle

Let us draw tangents t_1 and t_2 from the point Fc_1 to the conic a. As already shown earlier, straight lines t_1 and t_2 are perpendicular. We now choose an arbitrary point $T \sim f$ on the circle f and drop the tangent lines t^*_1 and t^*_2 from it to the conic a. Since the conic $e = \Pi_a(f)$ is the image of the circle f in polarity with respect to a, then the straight line $z = T^*_1 \circ T^*_2$ will be tangent to the conic e at the point Z. In turn, the point $Z = \Pi_a(t)$ where $t = \Pi_f(T)$. The points T and Z are on a common perpendicular tz dropped from T to z. Therefore, triangles $\triangle TT^*_1 Z$ and $\triangle TT^*_2 Z$ are right-angled, which proves the perpendicularity of the lines t^*_1 and t^*_2 regardless of the position of the point T. Thus, the circle f is the locus of the vertices of the rectangle circumscribed about the conic a (figure 7).

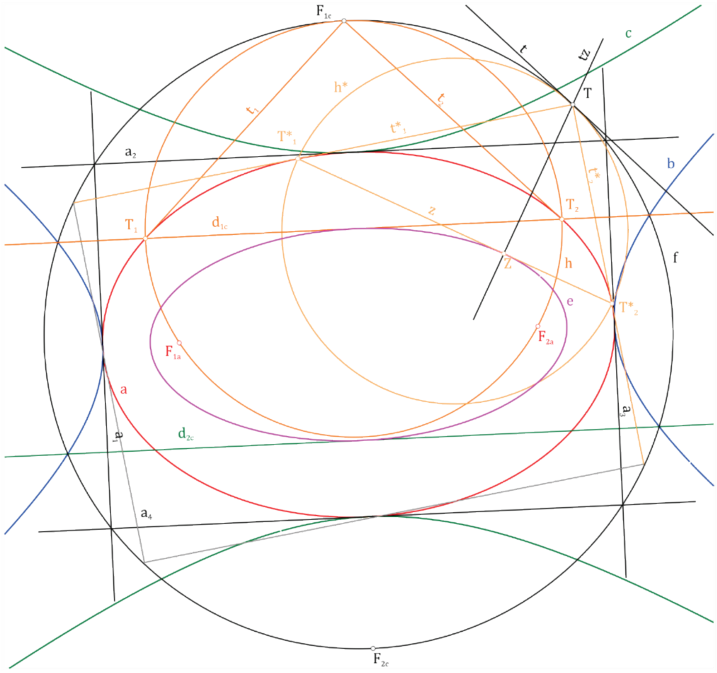

Figure 7: Some conics principal conjugations properties

Let the circle $w = F_1 \circ F_2$ be drawn through the foci of the conic a. The conic a induces an involution on its axis u, which arranges the points $B_1 - B_2$, $C_1 - C_2$, $D_1 - D_2$ ... into pairs. Taking the points of these pairs as diametrical, it is possible to construct circles $b = B_1 \circ B_2$, $c = C_1 \circ C_2$, $d = D_1 \circ D_2$ ... . It is obvious that pairs of points of the form $F_1 - F_1$ and $F_2 - F_2$ generate zero-circles in a series of circles with common centers P and Q, while moving a point V_1^∞ to infinity entails the coincidence of the corresponding point V with the center of the conic a. Considering the circle $v = V_1^\infty \circ V_2$ of the bundle as a circle of infinitely large radius, we come to the conclusion that the axes of the conic are not straight lines, but circles of a special form, decomposing into a pair of straight lines (figure 8). This assumption allows us to explain the presence of an "imaginary" diameter of the hyperbola, which does not intersect this curve at real points, since it is only a proper part of the degenerate circle, and its other part is an infinitely distant straight line that has real intersection points with the hyperbola. This geometric construction demonstrates the unified constructive nature of the foci of the second-order curve and its axes.

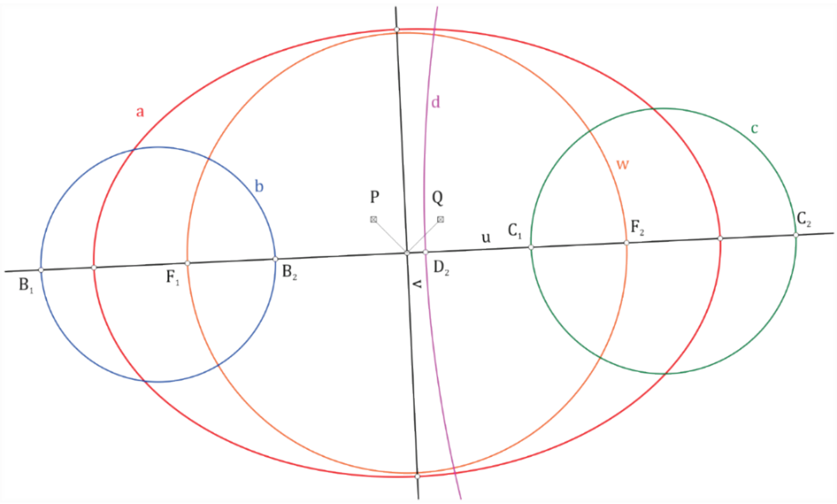

Figure 8: Representing the axis of a conic as a circle

3. Constructive relationship of principal conical conjugations with a line equidistant from two circles

Let there be two arbitrary circles a and b. Let's draw a straight line x through their centers Ca and Cb find the points $A_1, A_2 = a \times x$ and $B_1, B_2 = b \times x$. Let's define the involution $\xi \,|\, \begin{matrix} x; A_1; A_2 \\ x; B_2; B_1 \end{matrix}$ and find its center – the point M. This point is the homothetic center of the circles a and b, from which the common "external" tangents m_1 and m_2 are put onto these circles. Let's construct point\ $Ma_1 \sim m_1$, $Ma_1 \sim a$, $Mb_1 \sim m_1$, $Mb_1 \sim b$, $Ma_2 \sim m_2$, $Ma_2 \sim a$, $Mb_2 \sim m_2$,$Mb_2 \sim b$. Let's define another involution $\varsigma \,|\, \begin{matrix} x; A_1; B_2 \\ x; B_1; A_2 \end{matrix}$ and find its center – the point N. The point N is the second homothetic center of the circles a and b, from which the common "internal" tangents n_1 and n_2 are put onto these circles. Let's construct points $Na_1 \sim n_1$, $Na_1 \sim a$, $Nb_1 \sim n_1$, $Nb_1 \sim b$, $Na_2 \sim n_2$, $Na_2 \sim a$, $Nb_2 \sim n_2$, $Nb_2 \sim b$. Let us construct points $G_1 = m_1 \times n_2$, $H_1 = m_1 \times n_1$, $G_2 = m_2 \times n_1$ and $H_2 = m_2 \times n_2$ and draw circles $g_1 \sim Ma_1$, $h_1 \sim Nb_2$, $g_2 \sim Ma_2$ and $h_2 \sim Mb_2$ with centers at them, respectively. It should be noted that the radii of all these circles are the same. We define the point C as the fourth harmonic, based on the relation $(Ca, Cb, C, T^{\infty}) = -1$. Let's draw straight lines $am_1 \perp m_2$, $an_2 \perp n_2$, $an_1 \perp n_1$ through the point C. Note also that the line $am_2 = rxm_1$ is the radical axis of the pair of circles g_1, h_1, the line am_1 is the radical axis of the pair of circles g_2, h_2, the line an_1 is the radical axis of the pair of circles g_2, h_1, the line $an_2 = rxn_1$ is the radical axis of the pair of circles g_1, h_2. We construct points $Vm = an_1 \times n_1$ and $Vn = am_2 \times m_1$ and draw the circles $i \sim Vm$ and $j \sim Vn$ with the center at the point C. Having found the point $I_1, I_2 = i \times x$ s and $J_1, J_2 = j \times x$, we get the opportunity to construct straight lines $am_3 \sim I_1$, $am_3 \perp x$; $am_4 \sim I_2$, $am_4 \perp x$; $an_3 \sim J_1$, $an_3 \perp x$; $an_4 \sim J_2$, $an_4 \perp x$. Considering straight lines am_1, am_2, am_3, am_4 as asymptotes of a conic with known tangency points $am_3 \sim I_1$, $am_4 \sim I_2$, we construct a conic m, which is the first line equidistant from the original circles a and b. Considering straight lines an_1, an_2, an_3, an_4 as asymptotes of a conic with known tangency points $an_3 \sim J_1$, $an_4 \sim J_2$, we construct a conic n, which is the second line equidistant from the original circles a and b. Thus, the constructive relationship of lines equidistant from two given circles is shown, based on the analysis of the projective properties of the presented geometric scheme.

Knowing the conics m and n, we construct the system of their principal conjugations: $m - m^{*} - m^{**}$ and $n - n^{*} - n^{**}$, and also determine the foci and directrixes of all the obtained conics. It is easy to see that in the presented drawing, complementary conics m and n are confocal, therefore, their hyperbolic conjugations turn out to be confocal. This means that a common circle f passes through the focal points Fm_1, Fn_1, Fm_2, Fn_2, Fm^*_1, Fn^*_1, Fm^*_2, Fn^*_2, on which, among other things, there are points G_1, G_2, H_1, H_2 – the centers of the circles g_1, g_2, h_1, h_2. It is interesting, that $m^{**} \times n^{**} \times rab = R_1, R_2$, where rab is the radical axis of the circles a and b. Let us also pay attention to the fact that on figure 9 the ellipses n^{**}, m^{**}, have common tangents passing through the focal points of their principal conjugations.

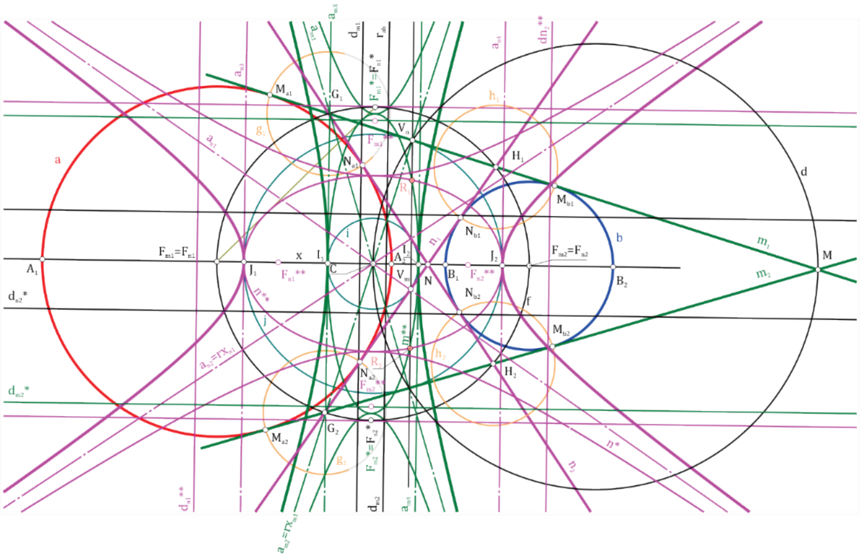

Figure 9: Projective scheme for constructing a line equidistant from two circles, option 1

On figure 10 is a drawing that implements the same construction, but on the condition that the circles a and b intersect at real points. In this case, one pair of tangents and the junction points become imaginary and cannot be directly displayed on the diagram. Despite this, the design scheme does not change, and its work leads to the desired result. In this case, the lines equidistant from the two circles are the confocal ellipse and the hyperbola. The two systems of principal conjugations of the two conics of the result are still in constructive interaction, but a number of previously presented properties do not appear in it. In particular, the ellipses and no longer have common tangents passing through the focal points of their principal conjugations, as a result of which this property has a local character and is not an invariant of the presented constructive scheme.

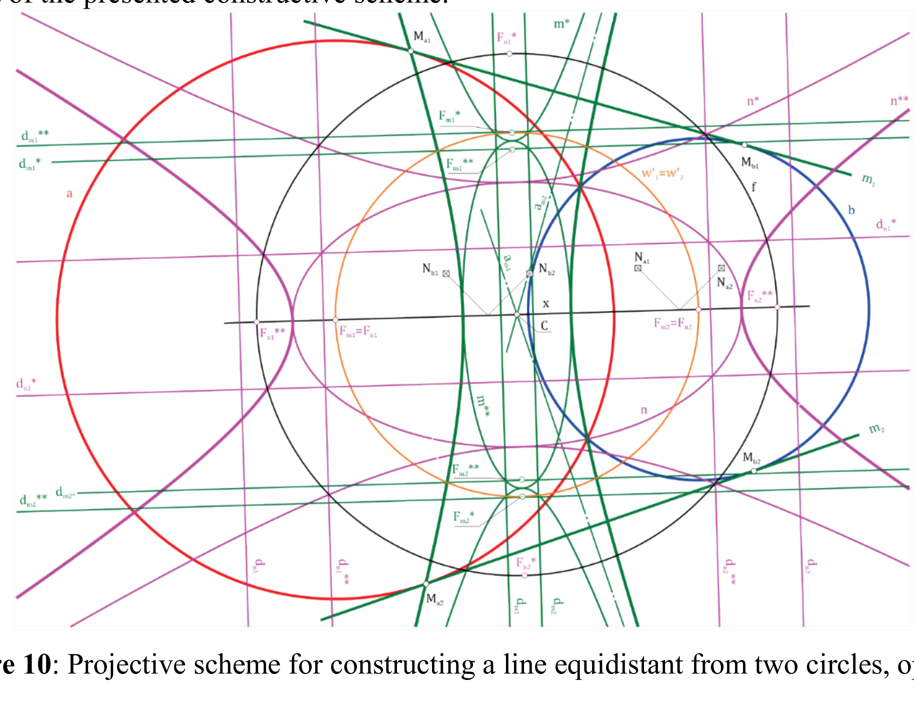

Figure 10: Projective scheme for constructing a line equidistant from two circles, option 2

4. Conclusion

The constructive geometric schemes considered in the article made it possible to draw the following conclusions:

1. Any conic is a component of a geometric image consisting of three conics forming mutual principal conjugations. This construction is the basis for proving many new geometric facts related to the nature of second-order curves.
2. A new substantiation of the geometric nature of the foci of second-order curves as centers of zero-circles of two mutually perpendicular bundles of circles is given.
3. A geometric substantiation of the nature of the axes of curves of the second order is given and the possibility of their inclusion as circles of a special kind is shown.
4. The constructive-geometric nature of the relationship between the foci and axes of a second-order curve is demonstrated.
5. A constructive geometric scheme is presented that demonstrates the projective properties of a line equidistant from two given circles.

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