Building 2D Low-Discrepancy Sequences for Hierarchical Importance Sampling Using Dodecagonal Aperiodic Tiling

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Abstract

This paper introduces a new method for building 2D low-discrepancy sequences and fast hierarchical importance sampling. Our approach is based on self-similar tiling of the plane with a set of aperiodic tiles having twelve-fold (dedecagonal) rotational symmetry. Sampling points of our low-discrepancy sequence are associated with tiles, one point per tiles. Each tile is recursively subdivided until the desired local density of samples is reached. A numerical code generated during the subdivision process is used for thresholding to accept or reject the sample. A special number system is specially tailored in order to allow linear numbering of the tiles. The resulting point distribution is more even, compared with that of popular Halton and Hammersley 2D low-discrepancy sequences. It can be successfully applied in a large variety of graphical applications, where fast sampling with good spectral and visual properties is required. Typical applications application are digital halftoning, rendering, geometry processing etc.

1 Introduction

Sampling is widely used in computer graphics. Hundreds of articles are devoted to studying important properties and limitations of sampling. Traditionally, Halton and Hammersley 2D low-discrepancy sequences are widely used for fast and efficient sampling of arbitrary functions defined on a 2D domain [Niederreiter 1992; Pharr and Humphreys 2004].

It is generally accepted today that sampling with blue-noise properties is preferable to others, for many reasons: for avoiding aliasing, for producing visually satisfactory artifact-free distributions, etc. [Ulichney 1987; Kollig and Keller 2003; Pharr and Humphreys 2004]. Very recently, a family of very fast techniques for the generation of blue-noise or Poisson-disc-like distributions have appeared [Ostromoukhov et al. 2004; Kopf et al. 2006; Dunbar and Humphreys 2006; Lagae and Dutré 2006]. All of them work in almost-linear time, with respect to the number of samples, with very low computational cost per sample. Each of the cited techniques has important advantages and limitations. Namely, boundary sampling [Dunbar and Humphreys 2006] and corner Wang tiling-based sampling [Lagae and Dutré 2006] do not offer any mechanism for smooth variation of the sample density as a function of arbitrary importance. Finally, recursive Wang tiling-based sampling [Kopf et al. 2006] produces a higher level of noise, compared to the others. Another important limitation of the latter technique consists in the large number of samples per tile (thousands, as presented in the paper). This is obviously an obstacle for hierarchical rendering algorithms, where the total number of samples is only a few dozens, and where the treatment of each sample, even rejected, has a cost.

The method presented in this paper is build upon the basic ideas of [Ostromoukhov et al. 2004]. The main difference of the proposed method is the nature of aperiodic tiling we use: instead of Penrose aperiodic tiling, we use here a derivation of the twelve-fold (dedecagonal) tiling called triangle-and-squares tilings, which is in tern a derivative of Solocolar's dodecagonal aperiodic tiling [Grünbaum and Shephard 1986; Socolar 1989]. As for the algorithm of adaptive importance sampling with the dodecagonal tiling, it is very close to that used in the cited article.

In this paper, we use a modified version of the original dodecagonal tiling. We build a dual of the triangle-and-squares tilings, as shown in Figure 1. The pentagons and the hexagons of the dual tiling form the basic tiles of our modified tiling. We build a set of original production rules for them, as it will be explained in the next section. It can be observed that the centers of the modified dodecagonal tiling (which are the vertices of the original triangle-and-squares tiling) do not move from one subdivition to the next one (see Figure 2-left). Also, it can be observed on Figure 2-right that the distribution of the sampling points achieved with our dodecagonal tiling is very even.

We take advantage of two important properties of dodecagonal aperiodic tiling. First, their construction is simple and deterministic, and their geometrical properties can be exhaustively studied. Second, dodecagonal tiling is fundamentally self-similar. Consequently, we can easily build a sampling system very close to the Penrose tiling-based one.

In summary, the **main contributions** of the paper consist in

- building explicit production rules for dodecagonal aperiodic tiling, dual to triangle-and-square tiling;
- building the number system which allows to assign to each sampling point a threshold value between 0 and 1, linearly distributed. The sampling point sets obtained with our method are evenly distributed;
- building 2D low-discrepancy sequences for hierarchical importance sampling based on dodecagonal aperiodic tiling; the sequences exhibit blue-noise spectral properties.

In the next section, we show the basic construction of dodecagonal aperiodic tiling. Then, in 3 we describe briefly the number system associated to the dodecagonal aperiodic tiling. In Section 4, we present the importance sampling algorithm. Finally, in Section 5, we draw some conclusions and discuss future work related to this article.

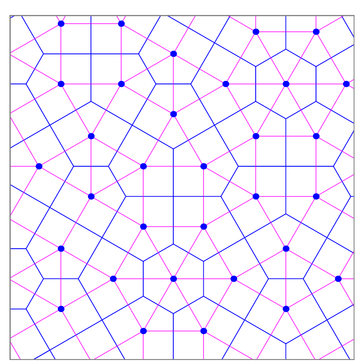

Figure 1: *Triangle-and-squares aperiodic tiling (red lines) and its dual (blue lines).*

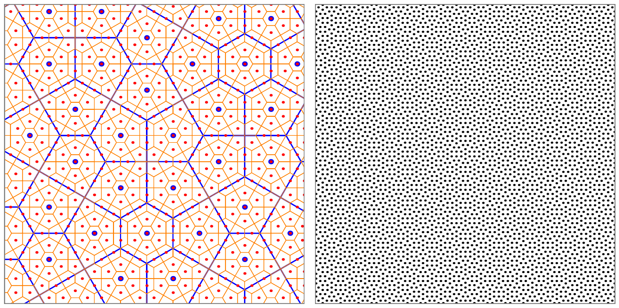

Figure 2: *(left) Two levels consecutive of subdivision) of our dodecagonal tiling, superimposed. (right) A typical distribution of dots obtained with our dodecagonal tiling.*

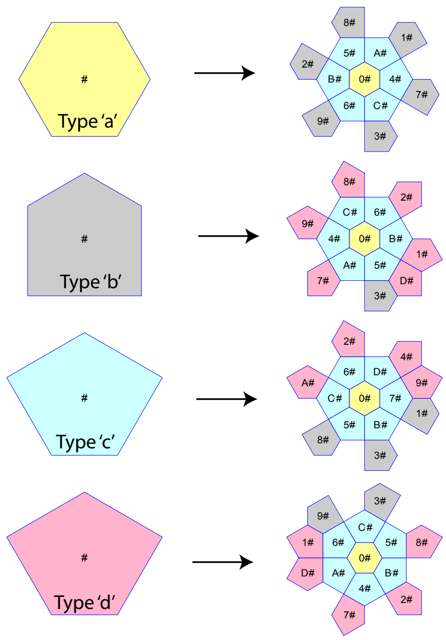

Figure 3: *Production rules used in dodecagonal aperiodic tiling.*

2 Basic Construction of Dodecagonal Aperiodic Tiling

Let us consider the dodecagonal subdivision process shown in Figure 3 as a recursive subdivision process. A special binary code called F-code (the term borrowed from [Ostromoukhov et al. 2004]) is assigned to each tile.

Out tiling is composed of four tiles of types 'a', 'b', 'c' and 'd'. This subdivision process can be described by the following production rules:

$$\mathscr{P}_{dodecagonal} := \begin{cases} a_{\#} \mapsto \{a_{0\#}, c_{4\#}, c_{5\#}, c_{6\#}, c_{A\#}, c_{B\#}, c_{C\#} \\ \quad b_{1\#}, b_{2\#}, b_{3\#}, b_{7\#}, b_{8\#}, b_{9\#}\}, \\ b_{\#} \mapsto \{a_{0\#}, c_{4\#}, c_{5\#}, c_{6\#}, c_{A\#}, c_{B\#}, c_{C\#} \\ \quad d_{1\#}, d_{2\#}, b_{3\#}, d_{7\#}, d_{8\#}, d_{9\#}, d_{D\#}\}, \\ c_{\#} \mapsto \{a_{0\#}, c_{5\#}, c_{6\#}, c_{7\#}, c_{B\#}, c_{C\#}, c_{D\#} \\ \quad b_{1\#}, d_{2\#}, b_{3\#}, d_{4\#}, b_{8\#}, d_{9\#}, d_{A\#}\}, \\ d_{\#} \mapsto \{a_{0\#}, c_{4\#}, c_{5\#}, c_{6\#}, c_{A\#}, c_{B\#}, c_{C\#} \\ \quad d_{1\#}, d_{2\#}, b_{3\#}, d_{7\#}, d_{8\#}, b_{9\#}, d_{D\#}\} \end{cases} \quad (1)$$

where x_y means a tile of type x having F-code y. The symbol '#' replaces the F-code of a tile before subdivision. Each subdivision left-concatenates one symbol in the range $[0,1,2,3,4,5,6,7,8,9,A,B,C,D]$ to the current F-code. Thus, after n subdivisions, the F-code will have the length of n symbols. F-codes can be interpreted as integer numbers according to the specially tailored number system explained in the next section. This number system is similar to the Fibonacci number system as described in [Knuth 1997] and [Graham et al. 1994].

Figure 6 shows the an example of application of our number system, starting with a sinfle tile of type 'a'.

3 Number System for Dodecagonal Aperiodic Tiling (δ- and D- number systems)

We build an original number system for the dodecagonal aperiodic tiling, based on ϕ- and F- (Fibonacci) number systems (See [Knuth 1997] and [Graham et al. 1994]).

First, let us recall the basics of ϕ- and F- (Fibonacci) number systems. The ϕ-system is a positional number system in base ϕ, where $\phi = \frac{1+\sqrt{5}}{2}$ is the *Golden Ratio*. Any real number x can be expressed in this system exactly as in our conventional binary or decimal systems, except that instead of using powers of two or ten, this system employs powers of ϕ. For example, the number $(101.001)_\phi$ in base ϕ is

$$(101.001)_\phi = \phi^2 + \phi^0 + \phi^{-3} \approx 3.8541_{10}$$

The ϕ-system is closely related to the F-system (the abbreviation for Fibonacci system). The F-system is also a positional system. Any integer n can be presented in the F-system as a sum of Fibonacci numbers F_j multiplied by their positional coefficients, which may be 0's or 1's. Thus, a number n can be expressed by its F-code $(b_m b_{m-1} \ldots b_3 b_2)_F$:

$$n = (b_m b_{m-1} \ldots b_3 b_2)_F \iff n = \sum_{j=2}^{m} b_j F_j. \quad (2)$$

The first index in the summation is $j = 2$ because of the convention used for Fibonacci numbers F_j:

$$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8, F_7 = 13, \ldots$$

The representation of numbers is not unique in the F-system, but it becomes unique if the rule of *normal form* is imposed: two adjacent 1's are not permitted. The procedure of conversion from an arbitrary sequence of 0's and 1's to the normal form, along with many other technical details, can be found in [Graham et al. 1994]. Here are the first twelve integers expressed in the F-system in normal form:

$$\begin{array}{lll} 1 = (00001)_F, & 2 = (00010)_F, & 3 = (00100)_F, \\ 4 = (00101)_F, & 5 = (01000)_F, & 6 = (01001)_F, \\ 7 = (01010)_F, & 8 = (10000)_F, & 9 = (10001)_F, \\ 10 = (10010)_F, & 11 = (10100)_F, & 12 = (10101)_F. \end{array}$$

In our dodecagonal tiling the linear scaling factor is $\lambda = 2 + Sqrt(3) \approx 3.73205$, therefore the area scaling factor is $\delta = (2 + Sqrt(3))^2 \approx 13.9282$. Like in ϕ-number system, any real number can be real number x can be expressed as the sum of powers of δ:

$$x = (a_m a_{m-1} \ldots a_0 a_{-1} \ldots a_{-n})_\delta \iff x = \sum_{j=-n}^{m} a_j \delta^j. \quad (3)$$

Similar to F-system, D-system allows to express any integer number in terms of its f-code:

$$n = (b_m b_{m-1} \ldots b_1 b_0)_D \iff n = \sum_{j=0}^{m} S_{b_j}^{(j)}. \quad (4)$$

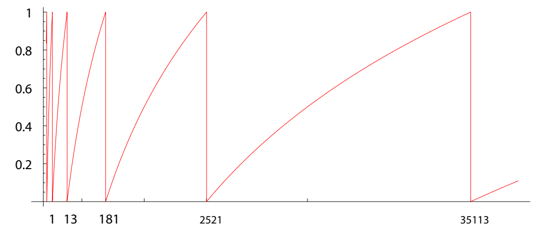

Figure 4: *Function* $\Psi(x)$.

where the sequences S_n are defined as follows:

$$S_n = \begin{cases} n * \sigma_1, & \textit{when } m < 2 \\ \sigma_1 + (n-1) * \sigma_2 & \textit{otherwise.} \end{cases} \qquad (5)$$

and where i-th term of the integer sequences σ_1 and σ_2 are defined as

$$\sigma_1^{(i)} = \frac{(\lambda_1 - 1)\lambda_1{}^{i-1} + \left(\frac{1}{\lambda_2} - 1\right)\lambda_2{}^{i}}{\lambda_1 - \lambda_2} \qquad (6)$$

$$\sigma_2^{(i)} = \frac{\lambda_1{}^{i} - \lambda_2{}^{i}}{\lambda_1 - \lambda_2} \qquad (7)$$

and the constants λ_1 and λ_2 are

$$\lambda_1 = (7 + 4 * Sqrt(3)), \quad \lambda_2 = (7 - 4 * Sqrt(3)).$$

Thus, for $i = 0, 1, 2, \ldots$, the integer sequences σ_1 and σ_2 are

$$\sigma_1 = \{1, 13, 181, 2521, 35113, 489061, 6811741, 94875313, \ldots\}$$

$$\sigma_2 = \{1, 14, 195, 2716, 37829, 526890, 7338631, 102213944, \ldots\}$$

Formulas (6) and (7) can be can be seen as an extension of the well-known Binet's formula for Fibonacci sequences.

The routine DODECATODECIMAL converts F-codes to the conventional decimal representation.

```
DODECATODECIMAL(fcode)
accumulator ← 0
for i ← 0 to LENGTH(fcode) − 1
    do
        index ← fcode[i]
        accumulator ← accumulator + S^(i)_index
return accumulator
```

Function $\Psi(x)$ that maps a real positive x onto interval $[0..1]$, as shown in Figure 4, is defined as follows:

$$\Psi(x) = (\log_\delta \cdot x) \ \mathbf{mod} 1$$

4 Importance Sampling with dodecagonal Aperiodic Tiling

This allows us to build an adaptive importance sampling system based on the dodecagonal subdivision system with the production rules (1). Our adaptive importance sampling system is simple. First, we cover the area of interest, where the importance is defined, with a tile of any type, say of type 'a', as shown in Figure 6. Then, we apply the recursive subdivision process according to the production rules (1). We stop subdividing when the required local subdivision level κ is reached. In this case, we output the coordinates of the center of each tiles, if the local importance is greater than the decimal value of the F-code of the current tile. Pseudo-code for this algorithm is as follows:

```
ADAPTIVESAMP(p of type tile)
        ▷ Structure tile contains the fields:
        ▷   type: determines type of subdivision
        ▷   LOS: Level of Subdivision
        ▷   refPoint
        ▷   v1, v2: tile's basis vectors
        ▷   fcode: used for computing threshold
local_LOS ← GETMAXLOSWITHINPOLYOMINO(p)
if p.LOS ≥ local_LOS
   then        ▷ Terminal: don't need more subdivisions
        local_importance ← GETLOCALIMPORTANCE(p)
        threshold ← GETTHRESHOLD(p.fcode)
        if local_importance ≥ threshold
          then        ▷ Output Selected Sample
                position ← p.refPoint
                OUTPUTSAMPLE(position)
        return
   else        ▷ Need more subdivisions
        {p1,..,p𝒜} ← SUBDIVIDEUSINGPRODUCTIONRULES(p)
        return {ADAPTIVESAMP(p1),..,ADAPTIVESAMP(p𝒜)}
```

Here, the function GETTHRESHOLD$(fcode)$ can be interpreted either as normalized version fo DODECATODECIMAL$(fcode)$ introduced earlier, or as a real fractional number in δ-system, in the range $[0..1]$

Importance density may be scaled by a factor *mag*, constant for the entire importance density image, in order to obtain the desired number of points. The required local level of subdivision κ can be determined as

$$\kappa = \lceil \log_\delta \max_{tile}(mag \cdot I(x,y)) \rceil, \qquad (8)$$

where $\lceil\ \rceil$ is the usual notation for ceiling, $I(x,y)$ is the importance value at position (x,y), and $\delta = \lambda^2 = 7 + 4 * Sqrt(3)$ is the tiling's area scaling factor. The value $\max_{tile}(\cdot)$ can be achieved with standard scan-conversion each tile. This scan-conversion is opened to possible optimization. If less precision is required but speed is capital, the importance can be tested only at a few points within the tile.

When compared to the popular Halton and Hammersley 2D low-discrepancy sequences, our 2D low-discrepancy sequence offers higher level of regularity, approaching blue noise property, as shown in Figure 7.

Applying dodecagonal importance sampling on a linear importance function produces very satisfactory result, shown in Figure 5.

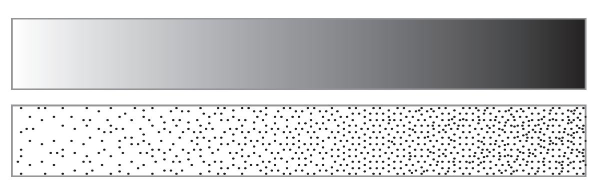

Figure 5: *A grayscale ramp (top) sampled with our method (bottom).*

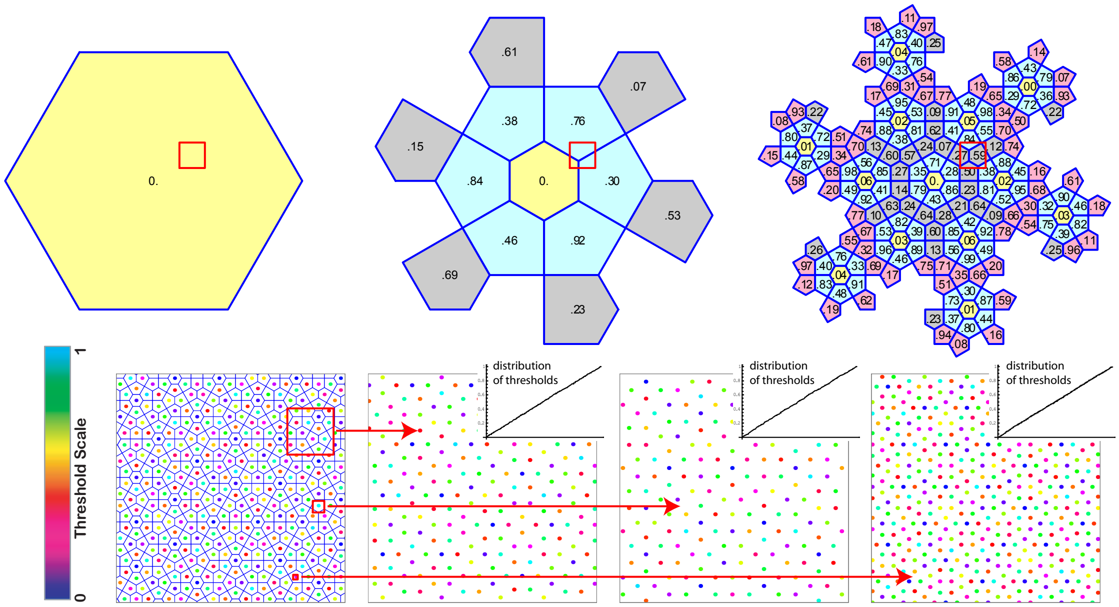

Figure 6: *(Top row) A tile of type 'a' subdivided two times, applying production rules (1). The normalized threshold is shown as a real number in the range* [0..1]*. (low row) Zooming into the red square of the top row, then subdividing and zooming into red squares of the leftmost distribution. It can be observed that any local area contains almost-linear distribution of threshold values in the range* [0..1].

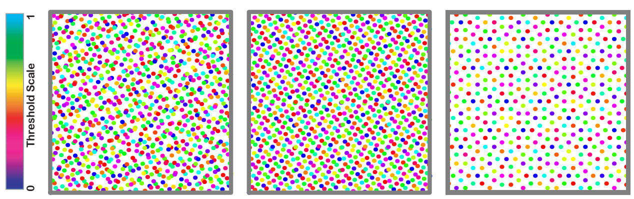

Figure 7: *(left) Halton, (center) Hammersley, and (right) dodecagonal low-discrepancy sequences of 2D points. Color designates threshold value in the range* [0..1].

5 Conclusion and Future Work

We have presented a new method for building 2D low-discrepancy sequences and fast hierarchical importance sampling, based on self-similar tiling of the plane with a set of aperiodic tiles having twelve-fold (dedecagonal) rotational symmetry.

The **main contributions** of the paper consist in (1) building explicit production rules for dodecagonal aperiodic tiling, dual to triangle-and-square tiling, (2) building the number system which allows to assign to each sampling point a threshold value between 0 and 1, linearly distributed, and (3) building 2D low-discrepancy sequences for hierarchical importance sampling based on dodecagonal aperiodic tiling; the sequences exhibit blue-noise spectral properties The sampling point sets obtained with our method are evenly distributed. Thanks to these features, the proposed method can be used in large variety of graphical applications, namely in digital halftoning, rendering (importance sampling), geometry processing etc. Typical examples of such applications can be found in [Ostromoukhov et al. 2004; Kopf et al. 2006; Dunbar and Humphreys 2006; Lagae and Dutré 2006].

In the future, we would like to better understand the limitations of the proposed method. Namely, we would like to explore the bias and aliasing artifacts that our method could eventually introduce in rendering. This will be the topic of future research.

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