COLOR RIDGES ON IMPLICIT POLYNOMIAL SURFACES

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Abstract

We define the ridges on a surface as surface curves where the principal curvatures take extrema along their corresponding principal directions. We classify the ridge points into four types (colors) according to some properties of the osculating spheres at the points. We propose an algorithm for the extraction of the ridge and its color on implicit polynomial surfaces. A formal algebraic computation is used to determine the equation of a surface whose intersection with the current surface gives the ridges.

1 Introduction

Shape features invariant under 3D rigid motions and scalings are important for many 3D shape processing operations including shape recognition, matching, and segmentation. The curvature tensor of a surface is invariant under all rigid transformations and leads to meaningful and intrinsic view- and scale-invariant features on every smooth surface.

Following [Porteous 1994] let us define the *ridges* on a smooth surface as curves consisting of the extrema of the principal curvatures along their corresponding principal directions.

Besides pure mathematical studies discovering and exposing the mathematical beauty of the ridges and associated structures [Koenderink 1990; Porteous 1994; Belyaev et al. 1997; Hallinan et al. 1999], the ridges and some subsets of them have been intensively studied in connection with research on the accommodation of the eye lens [Gullstrand 1904], structural geology [Ramsay 1967], image and data analysis [Yuille 1989; Eberly 1996], pattern recognition [Hallinan et al. 1999], machine vision [Kent et al. 1996], geographical information systems [Little and Shi 2001], quality control of free-form surfaces [Hosaka 1992] and other geometric modeling applications [Hartmann 1999], and computational anatomy [Pennec et al. 2000]. See also references therein.

Since the ridges are surface features of high-order differential nature (computations of them and their subsets may involve fourth-order surface derivatives), their robust detection on real-world data is a nontrivial task. In this paper, we propose an algorithm for detection of the ridge on implicit polynomial surfaces. Given an implicit polynomial surface, the result of our algorithm is another implicit polynomial surface whose intersection with the given one gives the ridges. We use Maple to implement the algorithm.

We test our method on implicit polynomial surfaces of third and fourth degrees. Of course, it is rather impossible to describe globally more or less complex shapes by low-degree polynomials. However low-degree polynomial approximations can be combined together by the partition of unity method in order to achieve a high quality approximations of shapes of arbitrary geometric and topological complexities [Ohtake et al. 2003]. Thus we consider our results as the first step towards developing a procedure for robust detection of the ridge curves and their subsets on real-world data.

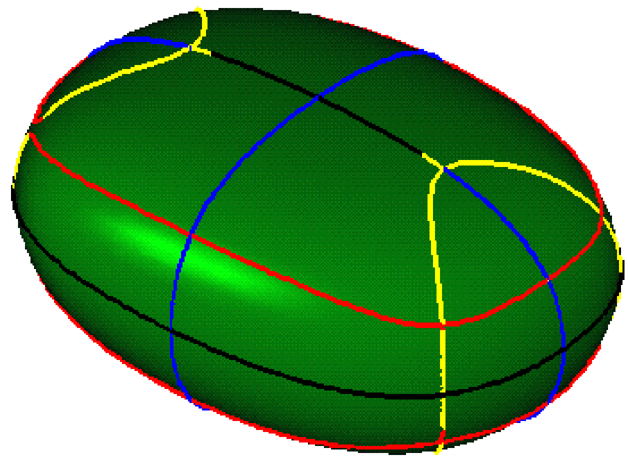

Figure 1: The color ridges on soap shape $x^4+2x^2y^2+4x^2z^2+4y^4+8y^2z^2+16z^4=1$

2 Rough classification of critical points

Following the Porteous' color notations [Porteous 1994] we distinguish four parts of the ridges on a generic surface using the classification of critical points of smooth functions [Arnold et al. 1993]. We paint these parts in red, blue, black, and yellow. For example, a given point of the surface is a red ridge point if in space there exists another point such that the distance from it has a degenerate minimum at the given point as a function on the surface. Similarly, if this distance has a degenerate maximum then the given point is a blue ridge of point. At isolated points ridge curves change their colors. We call these points *repaint*.

According to the classification of critical points (see §2 in [Arnold et al. 1993]), a typical degenerate critical point of a smooth function of two variables has one of the two forms

$$\pm s^2 + t^3 + \text{const}$$

in proper coordinates s, t.

If in proper new coordinates s, t a critical point has one of the following four forms

$$\pm s^2 \pm t^4 + \text{const},$$

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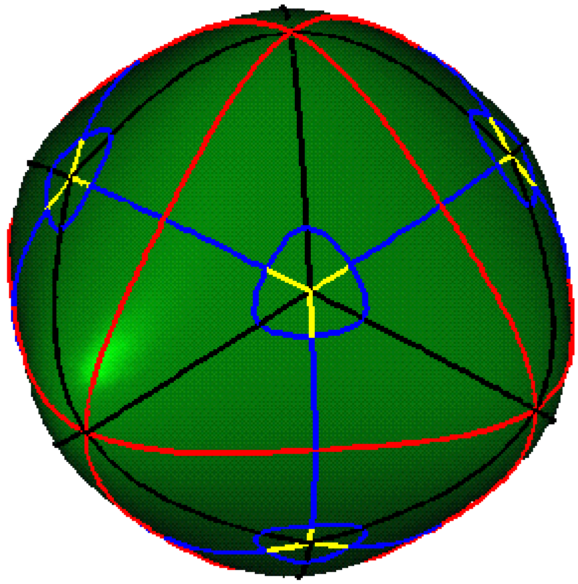

Figure 2: The color ridges on rounded octahedron $x^4 + 5x^2y^2 + 5x^2z^2 + y^4 + 5y^2z^2 + z^4 = 1$

then it is called *strongly* degenerate.

At last, we call the other degenerate critical points *complicated*. They do not have any of the above six forms in any coordinates.

3 Osculating spheres and color ridges

A sphere is said to *osculate* a surface at a point if it passes through the point and the distance from its center has a degenerate critical point at the given one as a function on the surface. If for a given point there exists an osculating sphere such that the surface is locally situated outside (inside) of it then the point is a *red* (*blue*) ridge point. In other words, the distance from the center of this sphere has a degenerate minimum (maximum) at the given point as a function on the surface.

We consider surfaces being generic either in the class of all smooth surfaces or in the class of symmetric ones with respect to several planes. A surface of this kind has two different osculating spheres at a typical point. In its neighborhood the distance from the center of each such sphere has the form

$$\pm s^2 + t^3 + R$$

in some coordinates s, t, where R is the radius of the osculating sphere. If the distance has either a strongly degenerate or a complicated critical point at the given one then the sphere is said to osculate *strongly*. The set of all such points is called the ridge points of the surface.

In particular, the points where the distance from the center of the strongly osculating sphere has the form

$$s^2 + t^4 + R$$

form the red ridge curves on the surface. The points where the distance from the center of the strongly osculating sphere

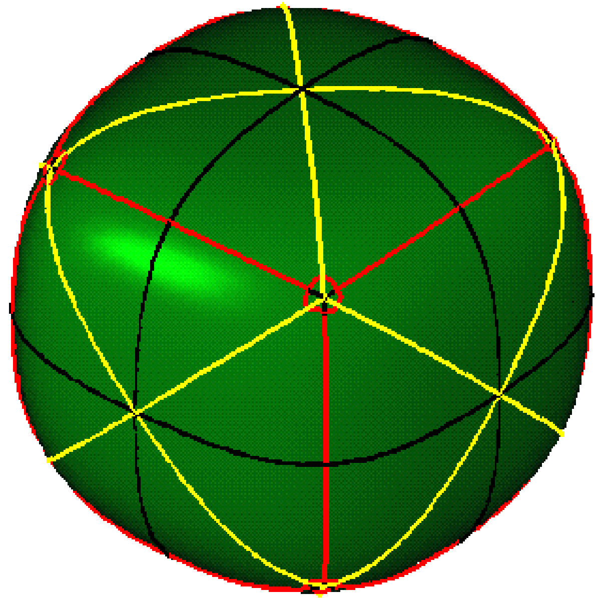

Figure 3: The color ridges on rounded cube $2x^4 + x^2y^2 + x^2z^2 + 2y^4 + y^2z^2 + 2z^4 = 1$

has the form

$$-s^2 - t^4 + R$$

form the blue ridge curves on the surface.

Besides, the *black* ridge curves consist of the points where the distance from the center of the strongly osculating sphere has the form

$$s^2 - t^4 + R.$$

The *yellow* ridge curves consist of the points where the distance from the center of the strongly osculating sphere has the form

$$-s^2 + t^4 + R.$$

4 Repaint points of the ridges

The color of the ridge curves on a generic surface can change at isolated points which we call *repaint*. Firstly, it happens at the points where the strong osculating sphere degenerates into a plane. They are called *godron* points (see [Kergosien 1991]). Secondly, a ridge curve changes its color when the distance from the center of the strongly osculating sphere has a complicated critical point. For example, any umbilic point is a repaint point. In this case the distance from the center of the strongly osculating sphere has one of the two forms $R + s^2t \pm t^3$ if the surface is generic in the class of all smooth surfaces.

5 Detection of the ridges

To get an equation of the ridge curves of an implicit polynomial surface we use the Boardman calculation of the Jacobian extensions of polynomial ideals described, for instance, in [Arnold et al. 1985, §2].

Let $F(\cdot,\cdot,\cdot) = 0$ be an implicit polynomial equation of a surface. Then

$$\Phi(\cdot,\cdot,\cdot; x, y, z; q) = 0$$

is an equation of a sphere ($q \neq 0$) or a plane($q = 0$) being tangent to the surface at the point (x, y, z), where

$$\Phi(u, v, w; x, y, z; q) =$$

$$= \frac{1}{2} q \left[(u-x)^2 + (v-y)^2 + (w-z)^2 \right] +$$

$$+ F_x(x, y, z)(u-x) + F_y(x, y, z)(v-y) + F_z(x, y, z)(w-z),$$

F_x, F_y, and F_z are the partial derivatives.

Let

$$\mathcal{A}(u, v, w; x, y, z; q) = J_{v,w}[\Phi, F(u, v, w)],$$

$$\mathcal{B}(u, v, w; x, y, z; q) = J_{w,u}[\Phi, F(u, v, w)],$$

$$\mathcal{C}(u, v, w; x, y, z; q) = J_{u,v}[\Phi, F(u, v, w)],$$

where the Jacobian of two functions f and g with respect to the variables u and v is the determinant of the matrix of their partial derivatives:

$$J_{u,v}[f, g] = f_u g_v - f_v g_u.$$

Similarly, the Jacobian of three functions f, g, and h with respect to the variables u, v, and w is the determinant of the matrix of their partial derivatives:

$$J_{u,v,w}[f, g, h] = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}.$$

It turns out that

$$\Phi \Big|_{u=x, v=y, w=z} = 0$$

since the sphere (or plane) passes through the point (x, y, z).

The tangency condition between the surface and the sphere (or plane) can be written in the form

$$\mathcal{A} \Big|_{u=x,v=y,w=z} = \mathcal{B} \Big|_{u=x,v=y,w=z} = \mathcal{C} \Big|_{u=x,v=y,w=z} = 0.$$

According to [Arnold et al. 1985], the following three conditions

$$J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{B}] \Big|_{u=x,v=y,w=z} = 0,$$

$$J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{C}] \Big|_{u=x,v=y,w=z} = 0,$$

$$J_{u,v,w}[\Phi, \mathcal{B}, \mathcal{C}] \Big|_{u=x,v=y,w=z} = 0$$

mean that our sphere (or plane) osculates the surface at the point (x, y, z). These equations are dependent and polynomial. Let $\mathcal{G}$ be their greatest common divisor.

If the osculation is strong then

$$J_{u,v,w}[\Phi, \mathcal{A}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{B}]] \Big|_{u=x,v=y,w=z} = 0,$$

$$J_{u,v,w}[\Phi, \mathcal{B}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{B}]] \Big|_{u=x,v=y,w=z} = 0,$$

$$J_{u,v,w}[\Phi, \mathcal{A}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0,$$

$$J_{u,v,w}[\Phi, \mathcal{C}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0,$$

$$J_{u,v,w}[\Phi, \mathcal{B}, J_{u,v,w}[\Phi, \mathcal{B}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0,$$

$$J_{u,v,w}[\Phi, \mathcal{C}, J_{u,v,w}[\Phi, \mathcal{B}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0.$$

So the following system of seven equations detects the ridge curves:

$$\mathcal{G} = 0, \tag{1}$$

$$J_{u,v,w}[\Phi, \mathcal{A}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{B}]] \Big|_{u=x,v=y,w=z} = 0, \tag{2}$$

$$J_{u,v,w}[\Phi, \mathcal{B}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{B}]] \Big|_{u=x,v=y,w=z} = 0, \tag{3}$$

$$J_{u,v,w}[\Phi, \mathcal{A}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0, \tag{4}$$

$$J_{u,v,w}[\Phi, \mathcal{C}, J_{u,v,w}[\Phi, \mathcal{A}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0, \tag{5}$$

$$J_{u,v,w}[\Phi, \mathcal{B}, J_{u,v,w}[\Phi, \mathcal{B}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0, \tag{6}$$

$$J_{u,v,w}[\Phi, \mathcal{C}, J_{u,v,w}[\Phi, \mathcal{B}, \mathcal{C}]] \Big|_{u=x,v=y,w=z} = 0. \tag{7}$$

These equations are polynomials of x, y, z, and q. Eliminating q we get a polynomial equation $H(\cdot, \cdot, \cdot) = 0$ of the ridges on our surface $F(\cdot, \cdot, \cdot) = 0$.

6 Maple implementation

A straightforward Maple program implements the resolution of the previous system of equations. The solution of this system is an equation of a surface H whose intersection with our surface determines the ridges.

The color plates illustrate the results we obtained with the Maple implementation on some implicit polynomial surfaces. The colors of the ridges correspond to those defined previously. The repaint points are not explicitly drawn since they correspond to changes of the color of the ridges and are thus easily detected. Note that there are no blue ridges on the rounded cube.

We compute the intersection between implicit surface H determined by the Maple program and the original surface F by a marching lines algorithm.

Quite often the equation of H can be decomposed into a set of planes (e.g. coordinate planes, planes of equation $x = y$, etc) and a remaining surface. Such a decomposition simplifies the detection of the ridges.

In the following Maple program, the ridge of the rounded octahedron is calculated.

Definition of Jacobians:

```
jac2:=proc(f,g,u,v);

RETURN(simplify(expand(diff(f,u)*diff(g,v)
                        -diff(f,v)*diff(g,u))));

end:

jac3:=proc(f,g,h,u,v,w)   local t1,t2,t3;

  t1:=diff(f,u)*diff(g,v)*diff(h,w)
                   -diff(f,u)*diff(g,w)*diff(h,v);

  t2:=diff(f,v)*diff(g,u)*diff(h,w)
                   -diff(f,v)*diff(g,w)*diff(h,u);
```

```
  t3:=diff(f,w)*diff(g,u)*diff(h,v)
                    -diff(f,w)*diff(g,v)*diff(h,u);

  RETURN(simplify(expand(t1-t2+t3)));

end:
```

Surface and derivatives:

```
F:=(u^4+v^4+w^4)+5*(u^2*v^2+u^2*w^2+v^2*w^2)-1:

cond:={u=x,v=y,w=z}:

subs(cond,F)=0;

Fx:=subs(cond,diff(F,u)):

Fy:=subs(cond,diff(F,v)):

Fz:=subs(cond,diff(F,w)):

Phi:=simplify(q*((u-x)^2+(v-y)^2
    +(w-z)^2)/2+Fx*(u-x)+Fy*(v-y)+Fz*(w-z)):

A:=jac2(F,Phi,v,w):

B:=jac2(F,Phi,w,u):

C:=jac2(F,Phi,u,v):

AB:=jac3(Phi,A,B,u,v,w):

AC:=jac3(Phi,A,C,u,v,w):

BC:=jac3(Phi,B,C,u,v,w):
```

Conditions of osculation:

```
D1:=subs(cond,AB):

D2:=subs(cond,AC):

D3:=subs(cond,BC):

G:=factor(gcd(D1,gcd(D2,D3)));
```

Conditions of strong osculation:

```
eq2:=subs(cond,jac3(Phi,A,AB,u,v,w)):

eq3:=subs(cond,jac3(Phi,B,AB,u,v,w)):

eq4:=subs(cond,jac3(Phi,A,AC,u,v,w)):

eq5:=subs(cond,jac3(Phi,C,AC,u,v,w)):

eq6:=subs(cond,jac3(Phi,B,BC,u,v,w)):

eq7:=subs(cond,jac3(Phi,C,BC,u,v,w)):
```

Elimination of q:

```
sw:=array([],1..7):

sw[1]:=G:

sw[2]:=numer(simplify(rem(eq2,G,q))):

sw[3]:=numer(simplify(rem(eq3,G,q))):

sw[4]:=numer(simplify(rem(eq4,G,q))):

sw[5]:=numer(simplify(rem(eq5,G,q))):

sw[6]:=numer(simplify(rem(eq6,G,q))):

sw[7]:=numer(simplify(rem(eq7,G,q))):

cd:=0:

for i from 2 to 7 do

  for j from 1 to (i-1) do

    cd:=gcd(cd,resultant(sw[i],sw[j],q)):

  od:

od:
```

Equation of ridges:

```
H:=factor(cd);
```

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