Two-dimensional Discrete Orthogonal Transforms with the «noise-like» Basis Functions

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Abstract

The generalization of the discrete orthogonal transforms with the basis functions generated in a pseudorandom way is the subject of the article. The examples of such transforms application in the field of videoinformation coding in the channels with the high level of «seldom» noise are also given.

Keywords: *Discrete orthogonal transforms, Image Compression, m-sequences.*

1. INTRODUCTION

Discrete orthogonal transforms (DOT)

$$\hat{x}(m) = \sum_{n=1}^{N-1} x(n) h_m(n) , \quad m = 0,1,...,N-1$$

are widely used in the fields of information coding, information transmission and discrete signals processing. Here $x(n)$ is the N-periodic input sequence, $\{h_m(n)\}_{m=0}^{N-1}$ are the basis functions orthogonal for some scalar (or Hermitian) product

$$< h_u, h_v >= \delta_{uv} .$$

In the real world channels the noise distorts the transmitted signals, especially those with the low numerical values. If the channel characteristics are such that some samples of the transmitted information can be irretrievably lost or strongly distorted regardless of their values then standard DOTs (Fourier, Hartley etc.) can hardly be used for information coding. The standard DOT properties depend on the correlation properties of the processed signal. Therefore the loss of some high amplitude components (for example Fourier components) during the videoinformation transmission results in a strong noise in the restored image. This noise has periodic structure for which the human vision is very sensitive. Another decision in this case is to code the information with the DOTs having such basis functions $h_m(n)$ for which the spectral components $\{\hat{x}(m)\}$ are «energetically equal». The notion of discrete M-transforms with the noise-like basis functions is introduced in [1]. The applications of such transforms for one- and two-dimensional information coding are considered in [2]-[4] (See also [5]).

Such transforms do not lead to energy concentration in a few spectral coefficients do not lower the redundancy attributed to the statistical relations between the elements of the signal to be transformed and efficiently remove insignificant information. Particularly in the process of compression of videoinformation after the inverse transformation was applied, the noise affecting the image is less notable for the human vision than in case of Walsh transforms. The basis functions of such transforms are based on the *m*-sequences (recurrent sequences of the finite field elements with the maximum periods) The sequences of this kind are widely used for the pseudorandom numbers generation, cryptography etc [6]-[7].

The M-transforms with the basis functions taking two different values with the (nearly) equal frequencies are considered in the publications mentioned above. In our article we consider the generalized M-transforms with the basis functions taking p different values. The one-dimensional transforms with the modified basis functions were announced in [8] and [9]. Some of the applications of such basis functions in image compression were considered in [10]. In this article we consider the two-dimensional versions of the proposed structures.

2. GENERALIZED ONE-DIMENSIONAL M-TRANSFORMS

Let F_p be the finite p-element field. Let $\varphi(n)$ be the r-order recurrent sequence

$$\varphi(n) = a_1\varphi(n-1) + ... + a_r\varphi(n-r) , \; a_j \in F_p \qquad (1)$$

with nontrivial initial values $(\varphi(0),...,\varphi(r-1))$.

<u>**Definition 1**</u> Let N be the period of sequence (1). If $N = p^r - 1$ then sequence (1) is called the m-sequence.

Using the slight modification of the corresponding proof Ref.[1] the following statement can be proven:

<u>**Proposition 1**</u> Let p be prime, $N = p^r - 1$. Let the numbers $A_0,...,A_{p-1}$ satisfy the following relation

$$A_k = k\frac{A_{p-1} - A_0}{p-1} + A_0, \; (k = 0,..., p-1)$$

Let the functions $h_m(n)$ be determined by the following relations

$$\begin{cases} h_0(n) = A_k & \text{if } \varphi(n) = k; \\ h_m(n) = h_0(m+n) \end{cases}$$

Then there exists the efficiently calculated constants A_0 and A_k such that the functions $\{h_m(n)\}_{m=0}^{N-1}$ form the orthonormal set

$$< h_u, h_v >= \sum_{n=0}^{N-1} h_u h_v = \delta_{uv}$$

Proof. Let us introduce the following notation

$$C = \frac{A_{p-1} - A_0}{p-1}, \; h_\tau(n) = h_0(n+\tau), \; A_0 = A$$

$$H_k(n) = \begin{cases} 1, \text{ if } \varphi(n) = k; \\ 0, \text{ if } \varphi(n) \neq k. \end{cases}$$

Then

$$h_\tau = \sum_{k=0}^{p-1} (A + kC) H_k(n+\tau) \qquad (2)$$

Let us take A_0 and A_{p-1} that the orthogonality condition for the set $\{h_m(n)\}$ is held in the form

$$< h_\tau, h_\nu >= N^{-1} \sum_{n=0}^{N-1} h_\tau(n) h_\nu(n) = \delta_{\tau\nu} \qquad (3)$$

Since the functions h_τ are obtained from each other by the cyclic shifts then the sum (3) depends only on $(\tau - \nu)$. Therefore we can consider only the case of $\nu = 0$.

Using (2) and (3) for $\tau = 0$ we get

$$N = \sum_{n=0}^{N-1} (\sum_{k=0}^{N-1} (A + Ck) H_k(n))^2 \qquad (4)$$

Since $H_i(n) H_j(n) = \delta_{ij}$ then using (4) we obtain

$$N = A^2 \sum_{i=0}^{p-1} \sum_{n=0}^{N-1} H_i(n) + 2AC \sum_{i=0}^{p-1} i \sum_{n=0}^{N-1} H_i(n)$$

$$+ C^2 \sum_{i=0}^{p-1} i^2 \sum_{n=1}^{N-1} H_i(n) =$$

$$= A^2 N + 2AC \sum_{i=1}^{p-1} i p^{r-1} + C^2 \sum_{i=1}^{p-1} i^2 p^{r-1} =$$

$$= A^2 (p^r - 1) + ACp^r (p-1) + C^2 p^r (p-1) \frac{2p-1}{6} \qquad (5)$$

Let $\tau \neq 0$. Then we have

$$0 = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (A + Ci)(A + Cj) S_{ij}(\tau) \qquad (6)$$

Here

$$S_{ij}(\tau) = \sum_{n=0}^{N-1} H_i(n) H_j(n+\tau) \,. \qquad (7)$$

The calculation of S_{ij} is the most difficult part of this poof. Using the standard method from number theory we can bring sum (7) to the trigonometric sum of the special kind.

Let ω be the p-th root of –1. Then $S_{ij}(\tau)$ can be expressed as

$$S_{ij}(\tau) = \frac{1}{p^2} \sum_{n=0}^{N-1} \cdot \left[\sum_{m_1=0}^{q-1} \omega^{(\varphi(n) - i) m_1} \cdot \sum_{m_2=0}^{q-1} \omega^{(\varphi(n+\tau) - j) m_2} \right] =$$

$$= \frac{1}{p^2} \sum_{n=0}^{N-1} \cdot \left[\sum_{m_1=0}^{q-1} \sum_{m_2=0}^{q-1} \omega^{\varphi(n) m_1 + \varphi(n+\tau) m_2} \cdot \omega^{m_1 i + m_2 j} \right] =$$

$$= \frac{1}{p^2} \sum_{m_1=0}^{q-1} \sum_{m_2=0}^{q-1} \left[\omega^{m_1 i + m_2 j} \cdot \sum_{n=0}^{N-1} \omega^{\varphi(n) m_1 + \varphi(n+\tau) m_2} \right] =$$

$$= \frac{1}{p^2} \sum_{m_1=1}^{q-1} \sum_{m_2=1}^{q-1} \left[\omega^{m_1 i + m_2 j} \cdot \sum_{n=0}^{N-1} \omega^{\varphi(n) m_1 + \varphi(n+\tau) m_2} \right]$$

$$+ \frac{1}{p^2} \sum_{m_2=1}^{q-1} \left[\omega^{m_2 j} \cdot \sum_{n=0}^{N-1} \omega^{\varphi(n+\tau) m_2} \right]$$

$$+ \frac{1}{p^2} \sum_{m_1=1}^{q-1} \left[\omega^{m_1 i} \cdot \sum_{n=0}^{N-1} \omega^{\varphi(n) m_1} \right] + \frac{1}{p^2} \cdot \sum_{n=0}^{N-1} 1$$

Since $\varphi(n)$ is the m-sequence then the sequences $\varphi(n) \cdot m_1 + \varphi(n+\tau) \cdot m_2$ are also the m-sequences for $m_1 \neq 0$ and $m_2 \neq 0$. Thus the above relation expression can be transformed into

$$S_{ij}(\tau) = -\frac{1}{p^2} \sum_{m_1=1}^{q-1} \sum_{m_2=1}^{q-1} \omega^{m_1 i} \cdot \omega^{m_2 j} - \frac{1}{p^2} \sum_{m_2=1}^{q-1} \omega^{m_2 j}$$

$$- \frac{1}{p^2} \sum_{m_1=1}^{q-1} \omega^{m_1 i} + \frac{1}{p^2} \cdot N$$

If $i = j = 0$ then

$$S_{ij}(\tau) = -\frac{1}{p^2} \sum_{m_1=1}^{q-1} \sum_{m_2=1}^{q-1} 1 - \frac{1}{p^2} \sum_{m_2=1}^{q-1} 1 - \frac{1}{p^2} \sum_{m_1=1}^{q-1} 1 + \frac{1}{p^2} \cdot N =$$

$$= \frac{1}{p^2} \cdot N + \frac{1}{p^2} - 1$$

If $i = 0$ and $j \neq 0$ or $i \neq 0$ and $j = 0$ then

$$S_{ij}(\tau) = -\frac{1}{p^2} \sum_{m_1=1}^{q-1} \sum_{m_2=1}^{q-1} \omega^{m_2 j} - \frac{1}{p^2} \sum_{m_2=1}^{q-1} \omega^{m_2 j} + \frac{1}{p^2} \cdot N =$$

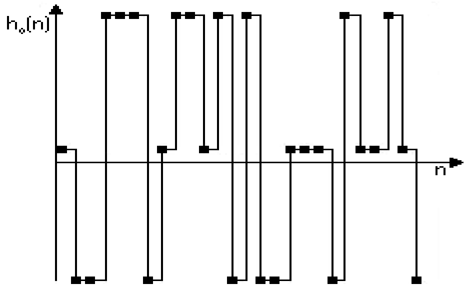
$$= \frac{1}{p^2} N + \frac{1}{p}$$

If $i \neq 0$ and $j \neq 0$ then

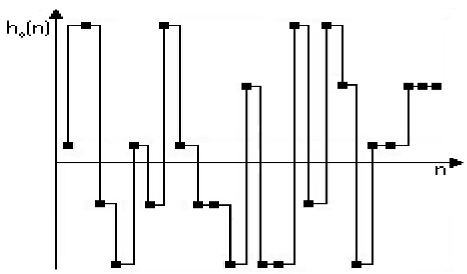
$$S_{ij} = \frac{1}{p^2} \cdot N + \frac{1}{p^2}$$

Substituting $S_{ij}(\tau)$ in (6) we get an explicit relation between A and C. This relation together with (5) brings the system of equations for determining A and C and therefore A_0 and A_{p-1}.

The examples of the basis function $h_0(n)$ for different p and N are shown on the Figure 1.



(a)



(b)

Figure 1:

Function $h_0(n)$ for (a) N=26, p=3, r=3; (b) N=24, p=5, r=2.

Definition 2 The transform (1) with the basis functions $\{h_m(n)\}$ defined in Proposition 1 is called the generalized M-transform (GM-transform).

3. TWO-DIMENSIONAL GM-TRANSFORMS

The one dimensional M-transforms introduced in the previous section can be used for two-dimensional digital arrays (images) coding after the standard digital image processing methods were applied.

The N×N pixel images can be represented by one-dimensional arrays in a number of ways (Figure 2).

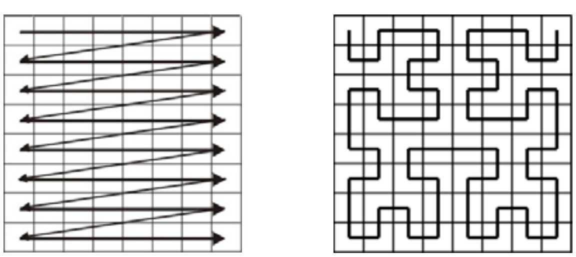


Figure 2: Different ways of representing two-dimensional $N\times N$ array as a one-dimensional N^2 – element array.

The two-dimensional 127×129 -points basis function for $p=2$ and $r=7$ is given on the Figure 3.

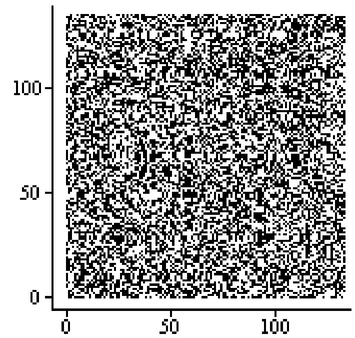


Figure 3.

(a) We can introduce the separable two-dimensional GM-transform by the following relation.

$$\hat{x}(m_1,m_2)=\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}x(n_1,n_2)h_{m_1}(n_1)h_{m_2}(n_2) \qquad (8)$$

In case (a) the two-dimensional N×N-points GM-transform calculation is reduced to the calculation of *one*-dimensional N^2-points GM-transform. In case (b) the *two*-dimensional N×N-points GM-transform calculation is reduced to the calculation of N one-dimensional N-points GM-transforms. This calculation is done using the standard "row-column" (cascade) scheme:

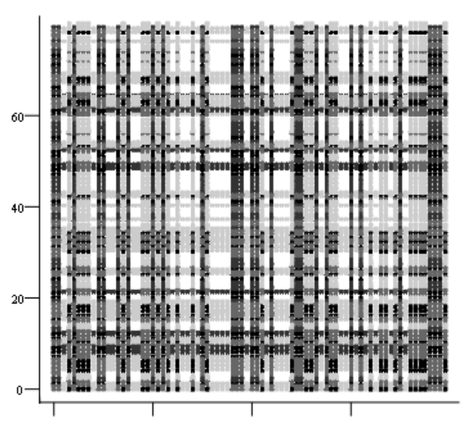
$$\hat{x}(m_1,m_2)=\sum_{n_2=0}^{N-1}\sum_{n_2=0}^{N-1}x(n_1,n_2)\chi_{m_1,m_2}(n_1,n_2)=$$

$$=\sum_{n_1=0}^{N-1}\left[\sum_{n_2=0}^{N-1}x(n_1,n_2)h_{m_2}(n_2)\right]h_{m_1}(n_1)=\sum_{n_1=0}^{N-1}Y(n_1,m_2)h_{m_1}(n_1)$$

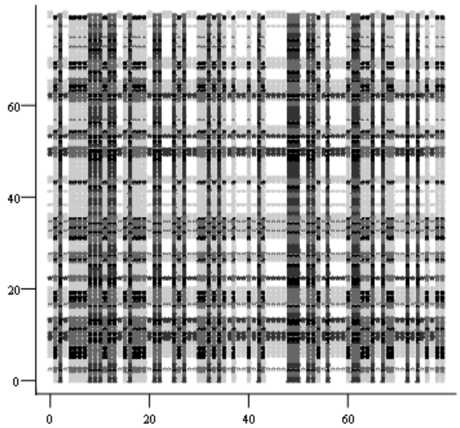
where

$$Y(n_1,m_2)=\sum_{n_2=0}^{N-1}x(n_1,n_2)h_{m_2}(n_2),\ \chi_{m_1,m_2}(n_1,n_2)=h_{m_1}(n_1)h_{m_2}(n_2)$$

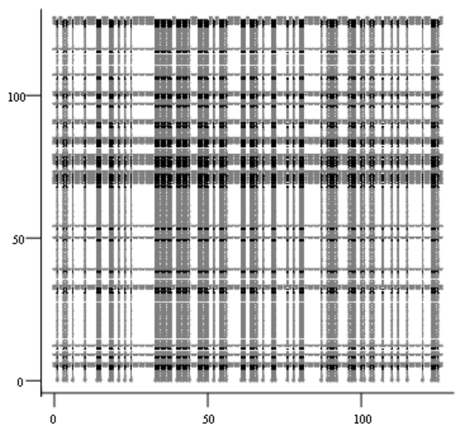
The examples of two-dimensional N×N-points basis functions for different values of p and r are given on the Figure 4.



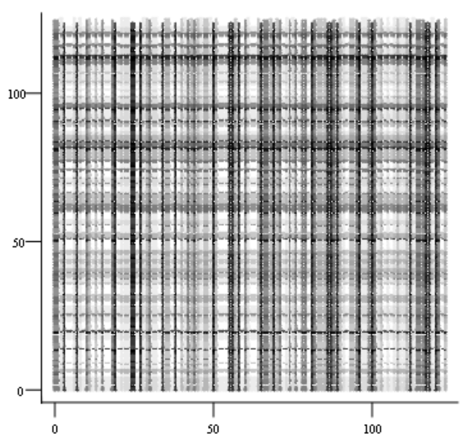
(a)



(b)



(c)



(d)

Figure 4:

(a) Function $\chi_{1,12}(n_1, n_2)$ for *N*=80, *p*=3, *r*=4;

(b) Function $\chi_{0,0}(n_1, n_2)$ for *N*=80, *p*=3, *r*=4;

(c) Function $\chi_{115,22}(n_1, n_2)$ for *N*=127, *p*=2, *r*=7;

(d) Function $\chi_{11,5}(n_1, n_2)$ for *N*=124, *p*=5, *r*=3.

4. FAST ALGORITHMS FOR GM-TRANSFORMS

The main property of GM-transforms is the existence of fast algorithms of their calculation.

Let us show that transforms (1) and (8) can be represented in a form of one- and two-dimensional convolution respectively.

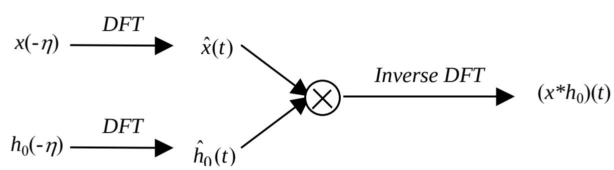
Let $\eta = -n,\ \eta_1 = -n_1,\ \eta_2 = n_2$.

The signal $x(n)$ and the functions $h_m(n)$ are N-periodic. Thus considering the introduced notation relations (1) and (8) can be rewritten in the form

$$\hat{x}(m) = \sum_{\eta=1}^{N-1} x(-\eta) h_0(m-\eta) = (x * h)(m) \qquad (9)$$

$$\hat{x}(m_1, m_2) = \sum_{\eta_1=0}^{N-1} \sum_{\eta_2=0}^{N-1} x(-\eta_1, -\eta_2) h_0(m_1 - \eta_1) h_0(m_2 - \eta_2) \qquad (10)$$

Array (9) can be calculated in a standard way using the discrete Fourier transform (DFT).



The drawback of this scheme is that $N = p^r - 1$ can hardly be factored as

$$N = p_1^{\alpha_1} \cdot \ldots \cdot p_t^{\alpha_j} \qquad (11)$$

The numbers $p_1, \ldots, p_t$ in (11) are primes.

The DFT calculation for $N = p_1^{\alpha_1} \cdot \ldots \cdot p_t^{\alpha_j}$ can be done using Good-Thomas decomposition [11]. According to it we have to calculate $p_j^{\alpha_j}$ -point DFT ($j = 1 \ldots\ldots t$).

In particular, to implement the M-transform of 255×257 image according to the scheme (a) we have used the (2^{16}-1) points DFT. Since

$$2^{16} - 1 = (2^8 - 1)(2^8 + 1) = 3 \cdot 5 \cdot 17 \cdot 257$$

According to the Good-Thomas decomposition this transform is reduced to the calculation of 3-, 5-, 17- and 257-points DFTs. We can calculate 3- and 5-point DFTs using Vinograd algorithm. The calculation of 17-points DFTs is reduced to the 16-points convolution according to the Rader scheme [11]. This convolution can be calculated using standard FFT. In the same manner the calculation of 257-points DFTs is reduced to the calculation of 256-points convolution which in turn can be calculated using the 256-points FFT.

To implement the M-transform of 255×255 pixels image according to the scheme (b) we have used the fast algorithms for (3×3), (5×5) and (17×17)-points two-dimensional Fourier transforms.

Another decision in this case is to calculate (9) and (10) using polynomial transform method [11].

The detailed discussion concerning the fast algorithms for calculation of (9) and (10) will be given on presentation.

5. EXPERIMENTAL RESULTS

Figures 5b-5d illustrate the reconstructed images after 70 of 256×256 spectral components have been replaced by zeroes for Hartley transform (b), Hadamard transform (c) and GM-transform (d). The original image is depicted on Figure 5a. The «lost» transforms were chosen in a random way.

(a)

(b)

(c)

(d)

Figure 5: (a) Original image; (b) Hartley transform; (c) Hadamard transform; (d) GM transform.

6. CONCLUSION

In authors' opinion the capabilities of GM-transforms are not limited to the examples given in the article. It is clear that GM-transforms can be used for signal processing not only in the frequency field but also in the time field. Such problems arise when processing (in particular, when interpolating) non-uniform sampling signals [12]-[13].

7. ACKNOWLEDGEMENTS

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