Representation of Linear Segment Voronoi Diagram by Bezier Curves*

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***Abstract* – A new method to describe the Voronoi diagram of a set of line segments is presented. We introduce a new description of a segment Voronoi diagram by a straight control graph of a set of elementary and rational Bezier curves of the first and second degree. Also, the distance between Voronoi edges and line segments called radial function is represented in the same manner by Bezier curves description.**

Keywords: Voronoi diagram, Voronoi edges, radial function, parabolic edges, Bezier curves, control graph

I. INTRODUCTION

Voronoi diagram (VD) for line segments is a well-studied and widely used geometric structure (Aurenhammer, 1991, Held, 2001, Karavelas, 2004). Particularly important is its application to the construction of skeletons of polygonal shapes, which is used in the image analysis and recognition (Drysdale, Lee, 1978, Kirkpatrick, 1979). There are known effective $O(n \log n)$ algorithms to construct VD for the general set of linear segments (Fortune, 1987, Yap, 1987) as well as for the sides of a simple polygon (Lee, 1982) or multiply-connected polygonal figures (Mestetskiy, Semenov, 2008).

Geometric construction of a segment VD is simple enough: it is a planar graph with straight-line and parabolic Voronoi edges (Fig. 1). Explicit expression of VD in the form of a geometric graph with coordinates of the vertices and curves of edges is required in many applications. In particular, it is used for the pruning of skeletons, or for the transformation of the object shape trough transforms of skeleton and radial function.

However, many authors point to the disadvantage of parabolic edges of VD. Parabolic edges are often replaced by piecewise linear polyline. Parabolic edges problem generates the tendency to handle structures having linear edges only. This idea is implemented in the concept of *straight skeleton* (Aichholzer, Aurenhammer, 1996). Sometimes reluctant to work with such edges indicated the primary motivation for the use of the straight skeleton (Tănase, 2004).

But the straight skeleton suffers from certain shortcomings, videlicet: complexity of mathematical definition, low computational efficiency, regularization complexity if noise effects are available.

In this paper, we propose a simple and robust method for segment VD describing in the form of a graph with straight edges.

1. The set of segment VD edges consists of the first and second order elementary and rational Bezier curves. This set we call *the compound Bezier curve* or *Bezier curve graph*.

2. A compound Bezier curve is described by its straight control graph, which is obtained from the dual graph of VD. The control graph consists of control polygons of Bezier curves.

Thus, to describe the segment VD, a straight-line control graph

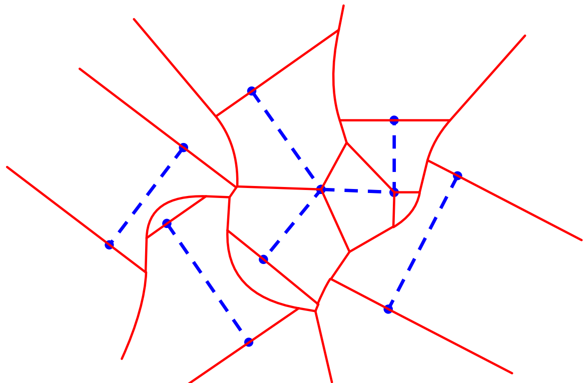

Fig. 1. The set of segments (dotted line) and its Voronoi diagram

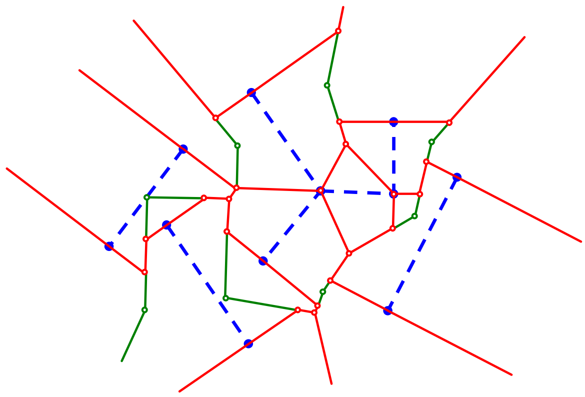

Fig. 2. The control graph of the Voronoi diagram of line segments (dotted line): line Voronoi edges (red) and control polygin edges (green)

is enough (Fig. 2). The set of control graph vertices consists of two subsets. The first subset is formed by Voronoi vertices of segment VD (they are depicted with red). And the second one consists of the certain control points called handles of Bezier curves (green). Straight edges of this graph are control polygons of Bezier curves. Some of them are described by the curve of the first degree (shown in red), and the other part - the curves of the second degree (green).

Usually algorithms for VD form the dual graph explicitly or implicitly. We show how based on this graph you can build a control graph that represents VD.

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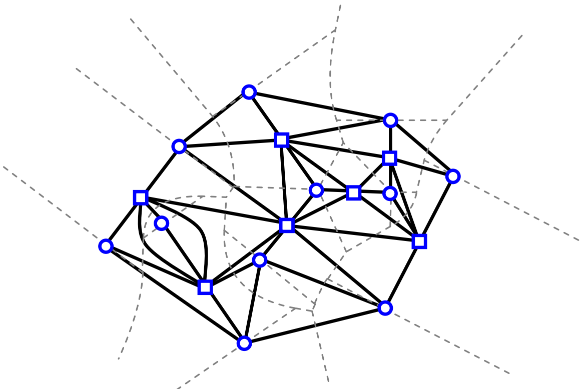


Fig. 3. Dual graph of Voronoi diagram from Fig.1. Vertex-sites depict circles, and segment-sites depict small squares

II. DEFINITIONS

Let's represent a set M of line segments as a finit set S of sites – points and open line segments (called *vertex-site* and *segment-site*). Endpoints of segments from M generate vertex-sites, and the segments without their endpoints generate segment-sites. Vertex-site and segment-site belonging to the same segment, called *neighboring* sites.

Thus, we distinguish three types of Voronoi edges. The first type (straight line) is defined by the pair "vertex-vertex", the second one (straight line) is defined by the pair "segment - segment" and the third one (parabola) is defined by the pair "vertex - segment". Every point of VD lies on these lines. Let us use the following terminology for Voronoi edges. There are *vv-bisectors*, *ss-bisectors* and *vs-bisectors* for the pairs of sites "vertex - vertex", "segment - segment" and "vertex - segment", respectively.

A *radial function* is determined at every point of VD. Radial function is equal to a euclidean distance between the point and its nearest site. The radial function at Voronoi edge assigns "the width of coridor" between two sites associated with this edge.

Adjacency graph or *Delaunay graph (DG)* of VD is an abstract graph whose set of vertices consists of the sites of VD, and the edge set contains all pairs of sites having adjacent Voronoi cells. DG is the dual graph of VD (Fig. 3).

DG defines the topological structure of VD. Each face of DG determines the Voronoi vertex. Each edge of DG determines the Voronoi edge. For visualization of DG, which is an abstract graph, we use the representation of DG as a geometric graph in the plane (Fig. 3).

We are going to build a representation of VD as a control graph (Fig. 2) based on the DG (Fig. 3), which is assumed to be known.

III. VORONOI VERTICES

The Voronoi vertices are equidistant to three or more sites. To find these vertices tangent circles can be constructed for the triplets of sites. Calculation of such circles involves a number of geometric tasks (Fig.4) related to the following combinations:

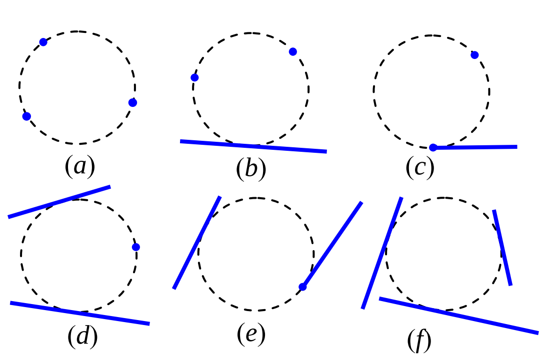


Fig. 4. Tangent circles for the triplets of sites

- three vertex-sites (Fig. 4*a*);
- two vertex-sites and one segment-site (Fig. 4 *b,c*);
- two segment-sites and one vertex-site (Fig. 4 *d,e*);
- three segment-sites (Fig. 4 *f*).

The second and third combinations involve different cases depending on whether triplet includes neighboring sites.

Assume that the tangent circle exists and the sequence of tangent points is defined. This condition holds for triples of vertices of each face of DG. Then the tangent circle is unique. To compute the center p of the circle tangent three sites s_1, s_2, s_3, the following system of equations is to be solved:

$$\begin{cases} d^2(p,s_1) = d^2(p,s_2) \\ d^2(p,s_1) = d^2(p,s_3) \end{cases}$$

In the cases in Fig. 4 *a,c,e,f* both equations are linear. But in the cases in Fig. 4 *b,d* one equation is linear, and the other is quadratic. After expressing the Y-coordinate of the point p through the X-coordinate in the linear equation it become possible to reduce the second equation to the usual quadratic equation, which is easily solved.

The obtained solution has to satisfy two auxiliary conditions, which are easily checked. The first condition requires the projections of p onto the segment-sites to lie on these segments themselves. The second condition requires the tangent circle to lie inside the figure. This means the center of tangent circle is required to lie to the left of the segment-site.

IV. VORONOI EDGES AS BEZIER CURVES

Explicit description of the parametric curve $V(t) = (x(t), y(t)),\ \ t \in [0,1]$ provides handy tools to deal with parabolic Voronoi edges. $V(t)$ determines the edge with Voronoi vertices $V(0)$ and $V(1)$. Also, a parametric representation is convenient to describe the radial function. Pair $\langle V(t), r(t) \rangle$ defines the edge of VD $V(t)$ and its "width" $r(t)$, which is a convenient form for medial representations of objects.

The basic idea is to represent the edges of VD and related radial function through conventional and rational Bezier curves.

First, we consider the representation of linear edges, which are *ss*-bisector. These edges are described by Bézier curves of the first degree

$$V(t) = V_0 B_0^1(t) + V_1 B_1^1(t), \quad t \in [0,1]. \qquad (1)$$

Here points V_0, V_1 denote endpoints of bisector. $B_0^1(t) = 1 - t$ and $B_1^1(t) = t$ are Bernstein polynomials.

Given the terminal points V_0 and V_1 of a linear *ss*-bisector together with radii r_0 and r_1 of the disks centered at V_0 and V_1, respectively, we can find the radius of the empty disk centered at any inner point of the edge $V_0 V_1$. It is obvious that the radius of empty disk centered at the point $V(t)$ is $r(t) = r_0 B_0^1(t) + r_1 B_1^1(t)$.

As shown in (Kim, 1995), a parabolic line could be represented as a quadratic Bezier curve

$$V(t) = V_0 B_0^2(t) + V_1 B_1^2(t) + V_2 B_2^2(t), \quad t \in [0,1], \quad (2)$$

where $B_0^2(t) = (1-t)^2$, $B_1^2(t) = 2t(1-t)$, $B_2^2(t) = t^2$ are *Bernstein polynomials*. This curve is determined by its *control triangle* $\{V_0, V_1, V_2\}$. The points V_0 and V_2 are called the *endpoints*, and the point V_1 is *handle point* of the Bezier curve.

Using this idea we represent a parabolic bisector of VD as a quadratic Bezier curve. Such a way of edge description is compact and easy-to-use since the only one handle point together with two endpoints defines every edge.

Consequently, in order to obtain *vs*-bisector as the Bezier curve it is necessary to calculate tangent lines at the endpoints of bisector and to find the intersection of these lines. Let us consider the solution of this problem.

Examine Voronoi edge V_0V_2 that is *vs*-bisector for the vertex-site A and the segment-site B (Fig. 5). Let r_0 and r_2 denote the circle radii at the vertices V_0 and V_2. For definiteness, we assume that when driving on the bisector from V_0 to V_2 the site A is to the left and the site B is to the right of the bisector. We choose a right-handed system, in which the axis Ox is parallel to the segment-site B and is equidistant from the sites A and B, while the axis Oy runs through the site A perpendicular to the site B. In this system, the following points have the coordinates $V_0 = (x_0, y_0)$, $V_2 = (x_2, y_2)$, $A = (0, c)$. The equation of the line on which B lies is $y = -c$, and the equation of the parabola with focus $A = (0, c)$ and directrix $y = -c$ has the form $x^2 - 4cy = 0$.

Bisector is a segment of a parabola between the points (x_0, r_0) and (x_2, r_2). It is described as quadratic Bezier curve (2), where V_0 and V_2 are Voronoi vertices, and a handle point of Bezier curve $V_1 = (x_1, r_1)$ is an intersection point of tangents lines of parabola in the end points $V_0 = (x_0, r_0)$ and $V_2 = (x_2, r_2)$. Therefore we can find V_1 through equations:

$$\begin{cases} 2x_0 \cdot x - 4c \cdot y = x_0^2 \\ 2x_2 \cdot x - 4c \cdot y = x_2^2 \end{cases}.$$

The solution of the system is

$$x_1 = \frac{1}{2}(x_0 + x_2), \; y_1 = \frac{1}{4c} x_0 \cdot x_2.$$

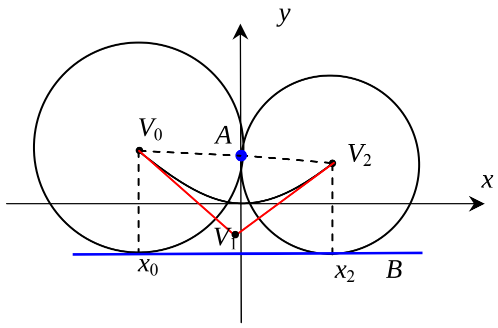


Fig. 5. Parabolic curve for *vs*-bisector

The radial function for *vs*-bisector can also be represented using parametric Bezier spline. In the local coordinate system (Fig. 5) we have simple relation between radii of disks and ordinates of the points of bisector $r(t) = y(t) + c$. From the property $B_0^2(t) + B_1^2(t) + B_2^2(t) = 1$ we obtain

$$r(t) = y_0 B_0^2(t) + y_1 B_1^2(t) + y_2 B_2^2(t) + c =$$
$$= (y_0 + c) \cdot B_0^2(t) + (y_1 + c) \cdot B_1^2(t) + (y_2 + c) \cdot B_2^2(t) =$$
$$= r_0 \cdot B_0^2(t) + (y_1 + c) \cdot B_1^2(t) + r_2 \cdot B_2^2(t).$$

If we set $r_1 = y_1 + c$ then

$$r(t) = r_0 B_0^2(t) + r_1 B_1^2(t) + r_2 B_2^2(t). \quad (3)$$

The disk centered at the handle point V_1 with radius r_1 is called a *handle disk*. As it follows from geometric analysis (Fig. 5), r_1 is equal to the distance from the point V_1 to the segment B.

Thus, equations (2) and (3) describe the shape and the radial function of *vs*-bisector.

Now let us consider the *vv*-bisector. We see that the radial function can not be presented by Bezier spline of first degree. Therefore, we will use more complex splines called rational Bezier curves to describe *vv*-bisector and its radial function.

Consider the edge V_0V_2 that is *vv*-bisector for a couple of vertex-sites A and B. Let r_0 and r_2 the circle radii at the vertices V_0 and V_2. We choose a right-handed system, which coincides V_0V_2 with the axis Ox and a positive direction is $\overrightarrow{V_0V_2}$ (Fig. 6*a*). The axis Oy passes through the sites A, B. In this system, the following points have the coordinates $V_0 = (x_0, 0)$, $V_2 = (x_2, 0)$, $A = (0, c)$ and $B = (0, -c)$. If r is the radius of the empty circle centered at the point x, then $r^2 - x^2 = c^2$. In the coordinate system Oxr this equation defines a hyperbola (Fig. 6*b*). We are interested only in the branch of the hyperbola, for which $r > 0$. As shown in (Kim, 1995) segment of the hyperbola between points (x_0, r_0) and (x_2, r_2) is described by the parametric equations of a rational Bezier curve

$$x(t) = \frac{x_0 \cdot B_0^2(t) + x_1 \cdot w_1 \cdot B_1^2(t) + x_2 \cdot B_2^2(t)}{B_0^2(t) + w_1 \cdot B_1^2(t) + B_2^2(t)}$$

$$r(t) = \frac{r_0 \cdot B_0^2(t) + r_1 \cdot w_1 \cdot B_1^2(t) + r_2 \cdot B_2^2(t)}{B_0^2(t) + w_1 \cdot B_1^2(t) + B_2^2(t)}.$$

Here (x_1, r_1) is the control point of the Bezier curve, which is the intersection of the tangents to the hyperbola at the points (x_0, r_0) and (x_2, r_2), i.e. is determined by solving the system of equations

$$\begin{cases} r_0 \cdot r - x_0 \cdot x = c^2 \\ r_2 \cdot r - x_2 \cdot x = c^2 \end{cases}$$

A weight coefficient w_1 is calculated as $w_1 = \dfrac{\lambda_1}{2\sqrt{\lambda_0 \lambda_2}}$ where $(\lambda_0, \lambda_1, \lambda_2)$ - the barycentric coordinates of the point (x^*, r^*) of hyperbola in the triangle with vertices (x_0, r_0), (x_1, r_1), (x_2, r_2). The point (x^*, r^*) can be selected between (x_0, r_0) and (x_2, r_2), for example, as $x^* = \frac{1}{2}(x_0 + x_2)$, $r^* = \sqrt{c^2 - (x^*)^2}$.

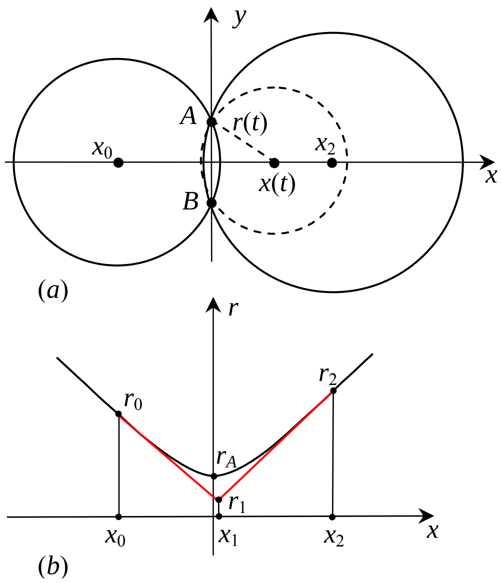


Fig. 6. Hiperbolic curve for *vs*-bisector

Coordinates $(\lambda_0, \lambda_1, \lambda_2)$ are calculated through the system of equations

$$\begin{cases} x_0 \cdot \lambda_0 + x_1 \cdot \lambda_1 + x_2 \cdot \lambda_2 = x^* \\ r_0 \cdot \lambda_0 + r_1 \cdot \lambda_1 + r_2 \cdot \lambda_2 = r^* \\ \lambda_0 + \lambda_1 + \lambda_2 = 1 \end{cases}.$$

These arguments show that on the basis of the computed parameters x_1, r_1, w_1 *vv*-bisector can be represented as a rational Bezier curve

$$V(t) = \frac{V_0 \cdot B_0^2(t) + V_1 \cdot w_1 \cdot B_1^2(t) + V_2 \cdot B_2^2(t)}{B_0^2(t) + w_1 \cdot B_1^2(t) + B_2^2(t)}, \quad (4)$$

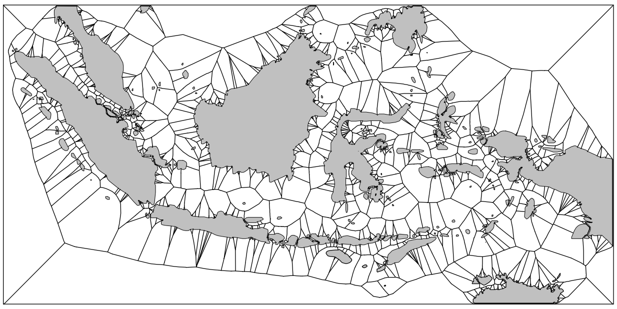
where $V_1 = V_0 \cdot (1 - \mu) + V_2 \cdot \mu$, $\mu = \dfrac{x_1 - x_0}{x_2 - x_0}$.

The corresponding radial function is defined as follows:

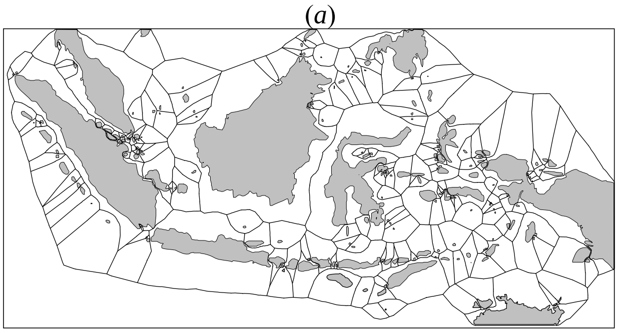
$$r(t) = \frac{r_0 \cdot B_0^2(t) + r_1 \cdot w_1 \cdot B_1^2(t) + r_2 \cdot B_2^2(t)}{B_0^2(t) + w_1 \cdot B_1^2(t) + B_2^2(t)}. \quad (5)$$

V. Control Graph of Compound Bezier Curve

Thus, VD is a union of Bezier curves of first and second order. These curves describe a connected geometrical graph. We call this graph the *compound Bezier curve* or *Bezier curve graph.* The segment VD can be represented by a straight-line geometric control graph of compound Bezier curve.



(*a*)



(*b*)

Fig. 7. VD of the Malay Archipelago: (*a*) polygonal figure and its internal skeleton, (*b*) VD after pruning

The advantage of Bezier curve graph is the ability to describing Bezier curves by their control polygons. Thus compound Bezier curve can be described by a graph composed of polygons elementary Bezier curves. We call it the *control graph of the segment VD*. Fig.2 shows a representation of compound Bezier curve by the control graph. Control graph has straight line edges. The set of control graph vertices includes all Voronoi vertices and handle points of quadratic Bezier curves.

The data structure describing control graph of VD includes feature for each edge type (*ss*-, *vs*-, *vv*-bisector). The edge *vs*-bisector has kept additional parameters: coordinates and radius of control circle (V_1, r_1). The edge *vv*-bisector has kept parameters (x_1, r_1, w_1).

VI. Implementation and Experiments

The proposed method for representing of VD used in the program of constructing of continuous skeletons for binary images, developed by the author (Mestetskiy, 1999, Mestetskiy, Semenov, 2008). The example (Fig. 7) presents the application of the method for constructing VD for a set of polygon-sites.

The source binary image has the size 1271×620 pixels and presents the Malay Archipelago. We need to construct a Voronoi diagram for the islands depicted on the map. First, we approximate this binary image by a multiply connected polygonal figure. We get the bounding box with 190 holes (islands) and 3728 vertices. Second, we get the Voronoi diagram for this polygonal shape and highlight its internal skeleton by pruning (Fig. 7*a*). And third, we cut all terminal branches of the skeleton and obtain the subgraph, which is VD of islands (Fig.7*b*). Some characteristics of this example are presented in the table. The total time for solving the problem, including the polygonal approximation, the construction of the skeleton and pruning is equal to 0.108 sec on the processor Intel 2.13 GHz.

	Skeleton	VD of islands
VD vertices	7822	4979
VD edges	8005	5162
ss-bisector	2822	1137
vs-bisector	3333	2479
vv-bisector	1850	1546

VII. Conclusions

In this paper, we have presented a new approach to describe the segment VD by stright line control graph of compound Bezier curves. One major advantage is the simplicity of this description. Another advantage is the independence from the algorithm of VD construction. Proposed form of segment VD representation gives the tool for storing and processing VD in geographical databases, computer graphics, and image processing systems.

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